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REAL BUSINESS CYCLES AND STATIONARY MARKOV EQUILIBRIUM WITH GLOBAL HETEROGENEITY

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REAL BUSINESS CYCLES AND STATIONARY MARKOV EQUILIBRIUM WITH GLOBAL HETEROGENEITY

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SUMÁRIO

RESUMO

Este artigo desenvolve um modelo de equilíbrio geral Markoviano estacionário com agentes heterogêneos, mercados completos e ausência total de fricções ou imperfeições, mas ainda assim capaz de gerar dinamicamente ciclos econômicos agregados de forma endógena. Em contraste com a literatura prévia, que depende de imperfeições de mercado (como custos de ajuste, restrições financeiras ou fricções informacionais) para explicar flutuações macroeconômicas, mostramos que comportamentos cíclicos podem surgir em economias plenamente eficientes.

O mecanismo decorre da interação endógena, mediada por preços, entre capital, consumo e fluxos de transações de ativos entre agentes com preferências e dotações diversas. Esses fluxos induzem movimentos persistentes de preços relativos que retroagem sobre as alocações agregadas, amplificando expansões e contrações coordenadas sem qualquer desvio da eficiência de Pareto.

Concluímos que políticas de estabilização pró-cíclicas ou anticíclicas, nesse tipo de ambiente, podem não apenas ser ineficazes, mas potencialmente reduzir o bem-estar, ao interferirem em alocações eficientes e ao risco de agravarem a dívida pública por meio de intervenções desnecessárias.

Palavras-Chave: Economia multissetorial; Equilíbrio recursivo; Modelos de crescimento; Espaço de estado mínimo; Ciclos Econômicos Reais.

ABSTRACT

This paper develops a stationary Markov general equilibrium model with heterogeneous agents, complete markets, and no frictions or imperfections, yet capable of generating aggregate business cycle dynamics endogenously. In contrast to prior literature that relies on market imperfections such as adjustment costs, financial constraints, or informational frictions to explain macroeconomic fluctuations, we show that cyclical behavior can emerge in fully efficient economies. The mechanism stems from the endogenous, price-mediated interaction of capital, consumption, and asset transaction flows across agents with diverse preferences and endowments. These flows induce persistent relative price movements that feed back into aggregate allocations, amplifying coordinated expansions and contractions without any deviation from Pareto efficiency. We conclude that pro-cyclical or counter-cyclical stabilization policies in such settings may not only be ineffective, but potentially welfare-reducing, as they interfere with efficient allocations and risk exacerbating public debt through unnecessary interventions.

Keywords Multi-sector economy, recursive equilibrium, growth models, minimal state space, Real Business Cycles.

JEL Code: C61 C62 C68 D50 D52 D58

1 Introduction

The dominant paradigm in modern business cycle theory explains aggregate fluctuations through dynamic general equilibrium models augmented by frictions or imperfections. These include time-to-build lags (Kydland and Prescott, 1982), sectoral technology shocks (Long and Plosser, 1983), incomplete markets (Backus et al., 1992), noisy measurements (Bomfim, 2001), price rigidity (Basu, 1995), constrained technology and network asymmetries (Acemoglu et al., 2012) or financial frictions (Oh and Yoon, 2020). While successful in reconciling model behavior with empirical volatility and persistence, these approaches invariably depend on the presence of some distortion to generate cyclical dynamics.

The time-to-build model of Kydland and Prescott (1982) demonstrates that aggregate fluctuations can arise in a frictionless neoclassical growth model once one introduces a multi-period construction process for capital goods. The assumption that productive capital becomes operational only after several periods creates internal propagation through delayed responses of investment and output to productivity shocks. While the allocation remains efficient, the friction lies in the temporal rigidity of capital formation. Similarly, Long and Plosser (1983) show that sector-specific technology shocks in an otherwise standard equilibrium model can generate correlated output movements across sectors, despite the absence of nominal frictions. However, their mechanism relies heavily on the structure of production linkages and the assumption that productivity shocks are uncorrelated but sufficiently persistent to induce aggregate effects.

A benchmark analysis of global market imperfections is presented in Backus et al. (1992), who construct a two-country real business cycle model with complete international financial markets. Despite allowing full risk-sharing across countries, the model produces counterfactually high consumption correlations relative to output correlations—known as the Backus-Kehoe-Kydland puzzle. This empirical failure highlights the implausibility of complete markets in international settings and suggests the existence of frictions such as limited enforceability or asset incompleteness. Their findings laid the foundation for subsequent work introducing endogenous incomplete markets to better reconcile theory with cross-country co-movement patterns.

An alternative channel of imperfection is explored in Bomfim (2001), who analyzes the macroeconomic consequences of noisy measurement in observed economic indicators. In his model, agents make optimal decisions based on imperfect signals about the underlying fundamentals, leading to aggregate fluctuations driven not by true shocks but by informational noise. This friction does not arise from distortions in preferences or constraints but from the inability of agents to observe the true state of the economy. The result is a misallocation of resources and a decoupling of real activity from fundamentals, generating a mechanism for real business cycle fluctuations rooted in signal extraction errors.

Acemoglu et al. (2012) propose a model where idiosyncratic firm-level shocks can lead to significant aggregate fluctuations due to input-output linkages in production networks. Contrary to the standard RBC assumption that such shocks cancel out, they show that network asymmetries—especially the presence of highly central firms—can amplify disturbances throughout the economy. A key feature of the model is that all inputs are used in fixed proportions, imposing a Leontief-type constraint on production and reinforcing the propagation mechanism. This framework offers a complementary explanation to traditional RBC models by emphasizing how network structure alone can generate aggregate volatility.

Financial and implementation frictions have also been explored as cycle generating mechanisms. Hairault et al. (1997) propose a framework with lags between innovation and the effective implementation of new capital, coupled with frictions in the labor market. Anticipated productivity changes influence current employment and investment decisions, but the delay in capital accumulation amplifies cyclical behavior. Oh and Yoon (2020) focus on the housing sector and embed real-options reasoning into a dynamic general equilibrium setting with credit constraints. The presence of financing frictions makes residential investment sensitive to future uncertainty, inducing agents to delay irreversible capital decisions. These frictions result in endogenous amplification and persistence of business cycles that deviate from the efficient benchmark.

In this paper, we propose an alternative theoretical route. We construct a stationary, recursive Markov general equilibrium model with global heterogeneity, complete markets, and no market imperfections. All allocations are Pareto efficient, and the environment features full insurance through state-contingent asset markets. Despite the absence of nominal rigidities, adjustment frictions, or information asymmetries, the model generates endogenous business cycles. The mechanism operates through persistent cross-sectional flows of capital, consumption, and financial claims among agents with distinct marginal propensities and state-contingent valuations. These intertemporal flows generate non-trivial dynamics in equilibrium prices—particularly asset prices and capital rental rates—which in turn reshape aggregate allocations persistently over time.

Our theoretical foundation builds on the insights of Duffie et al. (1994), who demonstrate the existence of non-degenerate stationary Markov equilibria in economies with heterogeneous agents and complete markets. We extend this framework based on Raad and Woźny (2019) and Raad et al. (2025) to show that such equilibria can give rise to recurrent aggregate fluctuations, even when the state space and agent fundamentals are stationary. These results highlight a fundamental source of business cycle dynamics that stems not from imperfections or policy shocks, but from the intrinsic structure of heterogeneous-agent interaction within an efficient allocation. Importantly, this has profound implications for macroeconomic policy: if business cycles are the natural result of efficient decentralized coordination, then stabilization interventions—whether counter-cyclical or pro-cyclical—may lead to large welfare losses and unnecessary fiscal burdens.

2 Model

In this paper, we adopt the notation introduced in Mas-Colell et al. (1995) and define the real vector space $\mathbb{R}^{\mathrm{n}}$ for each $\mathrm{n} \in \mathbb{N}$ endowed with the max norm topology[1] $|\cdot|$. Moreover, assume that each space of continuous functions is endowed with the sup norm. Let there be a finite set of agent types, denoted by $I = \{1, \cdots, \mathrm{i}\}$, and a finite set of firms $J = \{1, \cdots, \mathrm{j}\}$, partitioned into three sectors: capital production, capital rental, and consumption. These sectors are indexed by the sets J_k, J_κ, and J_ς, with respective cardinality defined by the vector $(\mathrm{j}_k, \mathrm{j}_\kappa, \mathrm{j}_\varsigma)$.

In the capital production sector, assume that capital is produced at the beginning of each period using two primary inputs: a raw material, with quantity denoted by $m^j \in M^j := \mathbb{R}_+$, and high-skilled

[1] That is, $|x| = \max\{|x_1|, \cdots, |x_{\mathrm{n}}|\}$.

labor, with quantity denoted by $l^j \in L^j := \mathbb{R}_+$. It is assumed that production plants do not allow for capital storage. The quantity of capital produced is denoted by $k^j \in K^j \subset \mathbb{R}_+$ with K^j convex for each $j \in J_k$, and it becomes available at the end of the production period.

In the consumption sector, a final consumption good is produced using capital input rented from the previous period, denoted by $\kappa^j \in K^j$, and low-skilled labor, denoted by $\ell^j \in L^j \subset \mathbb{R}_+$.

In the capital rental sector, firms are assumed to be endowed with physical capital (Hillebrand, 2012) and begin each period with an initial stock of capital available for rental, $k^j_{\_} \in K^j$. These firms engage in trading or "producing" rental contracts, where the contract value is measured in capital units[2] and denoted by $\kappa^j \in K^j$. Furthermore, at the end of each period, firms choose the total capital input $k^j \in K^j$ to be employed in the following period. Let γ be the fraction of capital remaining after depreciation for each firm $j \in J_\kappa$. Then, $1-\gamma$ denotes the proportion of capital that depreciates from the previously rented stock $k^j_{\_}$ for all $k^j_{\_} \in K^j$. Accordingly, the capital input for the next period, $k^j \in K^j$, comprises the residual capital $\gamma\kappa^j$ and the net adjustment $k^j - \gamma\kappa^j$, which may be non-negative (additional acquisitions) or non-positive (sales). The rented capital thus constitutes a contractual output linking the two capital sectors and represents the quantity of capital allocated in the previous period and made available for rental. In equilibrium, we will demonstrate that the total amount of capital used as an input in the consumption sector equals the total capital rented out in the previous period.

For simplicity, we assume that each consumer's choice set includes[3] quantities of capital rental, labor, raw material, *numéraire*, consumption good, and physical capital. Accordingly, we denote by $C^i \subset \mathbb{R}^7_+$ the full consumption set and assume that it is a closed box, with a typical element expressed as $c^i = (\kappa^i, \ell^i, l^i, m^i, \eta^i, \varsigma^i, k^i) \in C^i$ for each consumer $i \in I$. Similarly, the closed box production set of each firm[4] is written as $C^j \subset \mathbb{R}^7_+$, with a typical element $c^j = (\kappa^j, \ell^j, l^j, m^j, \eta^j, \varsigma^j, k^j)$ for each $j \in J$.

Consumers indirectly own shares in each firm through their endowments of linked assets, such as equities. There are j equities, with quantities[5] represented by the portfolio $\epsilon^i \in E^i \subset \mathbb{R}^{\mathrm{j}}$, which are exclusively traded by consumers. For each agent i, a typical portfolio choice is written as $\epsilon^i = (\epsilon^i_1, \epsilon^i_2, \cdots, \epsilon^i_{\mathrm{j}})$, where $\epsilon^i \in E^i$. The equities are in unitary net supply, that is, $\sum_{i\in I} \epsilon^i = 1 \in \mathbb{R}^{\mathrm{j}}$.

The set of equity prices is denoted[6] by $Q \subset \mathbb{R}^{\mathrm{j}}$, with a typical element represented as a row matrix[7] $q = [q_1, q_2, \cdots, q_{\mathrm{j}}]$. Note that in the capital production and consumption sectors, the equity price corresponds to the total firm value, whereas in the capital rental sector, it reflects the value of a single unit of capital. Let $p = [p_\kappa, p_\ell, p_l, p_m, p_\eta, p_\varsigma, p_k]$ be the row vector of unitary prices for intermediate and final goods[8]. Define $P \subset \mathbb{R}^7_{++}$ as the set of such price vectors p, representing prices for capital rental, labor, raw material, *numéraire*, consumption goods, and capital, respectively. Denote this set as the product space $P = P_\kappa \times P_\ell \times P_l \times P_m \times P_\eta \times P_\varsigma \times P_k$. Additionally, we normalize the *numéraire* price to one, i.e., $p_\eta = 1$. For all $i \in I$, define $X^i = C^i \times E^i$ and recall that the set of firms' choices is

[2] We will see that at the optimal choice $\kappa^j = \bar{k}^j_{\_}$.

[3] We assume that intermediary goods do not generate utility for the consumer.

[4] Assume that firms do not produce the *numéraire* good.

[5] Each equity also distributes profits according to a *pro-rata* rule.

[6] Since we consider the case of zero profits at the steady-state equilibrium, equities may exhibit negative prices outside of the steady state.

[7] We denote a row matrix by $[\cdots]$ and a column matrix by $(\cdots)$.

[8] That is, the non-saving goods.

given by C^j for all $j \in J$. A typical element of X^i is expressed as a column matrix $x^i = (c^i, \epsilon^i)$.

Exogenous uncertainty is captured by a finite set Z and a Markov transition function $\mu : Z \to \mathrm{Prob}(Z)$. Physical capital is interpreted as a real asset that enables intertemporal wealth transfers. Define the set of asset allocations as $A = K^{J_\kappa} \times E$, where $K^{J_\kappa} = \prod_{j \in J_\kappa} K^j$ with a typical element denoted by $\bar{a}_- = (\bar{k}_-, \bar{\epsilon}_-)$, where $\bar{\epsilon}_- = [\bar{\epsilon}^1_-, \cdots, \bar{\epsilon}^{\mathrm{i}}_-]$ and $\bar{k}_- = [\bar{k}^1_-, \cdots, \bar{k}^{\mathrm{j}_\kappa}_-]$, representing the previous period's asset distribution. Define the state space as $S = A \times Z$[9] with a typical element written as $s = (\bar{a}_-, z)$ and denote by $\widehat{P}$ the set of all continuous functions $\hat{p} : S \to P$. Let $\widehat{C}$ be the space of all continuous functions $\hat{c} : S \to C$, representing the transition of optimal consumption choices, and let $\widehat{A}$ be the space of all continuous functions $\hat{a} : S \to A$, representing the evolution of asset allocations.[10] Finally, define the set of price and profit transition functions as $\Gamma = \widehat{P} \times \widehat{Q} \times \widehat{Q}$.

Notation 2.1. We denote a row matrix as $[\cdots]$ and a column matrix as $(\cdots)$. Define the symbol without upper index as the Cartesian product over all agents for any allocation variable. For instance, write $C = \prod_{\iota \in I \cup J} C^\iota$ with a typical element as a row matrix $c = [c^\iota]_{\iota \in I \cup J}$. Moreover, when it is necessary to specify the cartesian product we write the index set in the upper index. For instance, write $C^I = \prod_{i \in I} C^i$. The remaining variables have an analogous interpretation. Define analogously the symbol without upper index for functions. Given $\{\nu, \nu'\} \subset \mathbb{R}^N$ and $M \subset N$, write $\nu^{M+} = \sum_{\mu \in M} \nu^\mu$ and $\nu\nu' = \sum_{\eta \in N} \nu_\eta \nu'_\eta$. We state that $\nu \geq \nu'$ when $\nu_\nu \geq \nu'_\nu$ for all $\nu \in N$. Given a random vector $\hat{z} : Z \to \mathbb{R}^{\mathrm{n}}$ define

$$\mathrm{e}[\hat{z}(\,\cdot\,)|z] = \sum_{z' \in Z} \hat{z}(z')\mu(z, z') \text{ for all } z \in Z.$$

We denote by 0 as the column vector $(0, 0, \cdots, 0) \in \mathbb{R}^{\mathrm{n}}$ or a null function. Write $\mathrm{tr} : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}}$ as the transpose function, that is, $\mathrm{tr}([y_1, \cdots, y_{\mathrm{n}}]) = (y_1, \cdots, y_{\mathrm{n}})$. Finally, denote generically by $\mathring{z}_\eta = (0, \cdots, 1, \cdots, 0) \in \mathbb{R}^{\mathrm{n}}$ as the unitary variation on the coordinate η around some point in $z \in \mathbb{R}^{\mathrm{n}}$.

2.1 Firms

Heterogeneous firms produce the consumption and capital goods in two sectors. Each sector has a large number of firms. Technology in each sector $j \in J$ is represented as the transformation function $f^j : C^j \times Z \to \mathbb{R}^2_+$ with $f^j = (f^j_\varsigma, f^j_k)$ for all $j \in J$. Therefore, for each $j \in J$ the correspondence of transformation constraint $\tilde{b}^j : K^j \times Z \rightrightarrows C^j$ is defined for each $j \in J$ by

$$\tilde{b}^j(k^j_-, z) = \begin{cases} \{c^j \in C^j : f^j_k(c^j, z) \geq k^j\} \text{ if } j \in J_k \\ \{c^j \in C^j : f^j_\varsigma(c^j, z) \geq \varsigma^j\} \text{ if } j \in J_\varsigma \\ \{c^j \in C^j : \kappa^j \leq k^j_-\} \text{ if } j \in J_\kappa \end{cases}$$

Assume that the *numéraire* is a type of transaction technology provided by a central authority and hence it is not produced in the markets. Furthermore, suppose to simplify that primary inputs are not produced.

[9] This set contains the minimal state space. We exhibit the minimal state space further ahead.

[10] Note that we use the symbol "$\widehat{\ }$" to denote the space of functions from S to the specified set or an operator in this space. Moreover, we use the symbol "$\widetilde{\ }$" to denote functions defined outside the state space S.

Recall that, in the capital sector, firms employ $(l^j, m^j) \in L^j \times M^j$ as inputs and exclusively produce the amount k^j of capital and hence $f^j_\varsigma = 0$ for $j \in J_k$. Under this condition we then define current net profits[11] $\tilde{\pi}^j : C^j \times P \to \mathbb{R}$ of any allocation $c^j \in \tilde{b}^j(k^j_-, z)$ for a price $p \in P$ as

$$\tilde{\pi}^j(c^j, p) = p_k k^j - p_l l^j - p_m m^j \text{ for all } (c^j, p) \in C^j \times P. \tag{1}$$

Analogously, in consumption sector firms only employ $(\kappa^j, \ell^j) \in K^j \times L^j$ as inputs and exclusively produce the amount ς^j of consumption goods and hence we have $f^j_k = 0$ for all $j \in J_\varsigma$. Thus define $\tilde{\pi}^j : C^j \times P \to \mathbb{R}$ as

$$\tilde{\pi}^j(c^j, p) = p_\varsigma \varsigma^j - p_\kappa \kappa^j - p_\ell \ell^j \text{ for all } (c^j, p) \in C^j \times P. \tag{2}$$

The capital rental sector solely engages in the transaction of capital goods and rental contracts, with no technological constraints on contract issuance. Therefore, we consider $f^j = 0$ for all $j \in J_\kappa$. If $\kappa^j \leq k^j_-$ represents the amount of capital available to rent at the beginning of each period then firms obtain $p_\kappa \kappa^j \geq 0$ as revenue at the end of each period. Moreover, it trades the variation[12] $k^j - \gamma\kappa^j \in \mathbb{R}$ of capital at the end of each period. Indeed, the remaining capital free from depreciation, combined with the traded capital at the end of each period, is limited by the total available capital for that period. Therefore, current profits $\tilde{\pi}^j : C^j \times P \to \mathbb{R}$ as are given by

$$\tilde{\pi}^j(c^j, p) = p_\kappa \kappa^j - p_k(k^j - \gamma\kappa^j) \text{ for all } (c^j, p) \in C^j \times P \tag{3}$$

Suppose that firms managers in capital rental sector have a type of “volatility” aversion such as the mean-variance preferences, or similarities. More specifically, we assume that the volatility expectation function $\tilde{\nu}^j : K^j \times S \times \widehat{P} \times \widehat{A} \to \mathbb{R}$ represents this behavior for $j \in J_\kappa$. Define $\tilde{u}^j : C^j \times S \times \widehat{P} \times \widehat{A} \to \mathbb{R}$ as

$$\tilde{u}^j(c^j, s, \hat{p}, \hat{a}) = \tilde{\pi}^j(c^j, \hat{p}(s)) - \tilde{\nu}^j(k^j, s, \hat{p}, \hat{a}) \text{ for all } (c^j, s, \hat{p}, \hat{a}) \in C^j \times S \times \widehat{P} \times \widehat{A}.$$

Given a transition price $\hat{p} \in \widehat{P}$, and a continuous function $\tilde{v}^j : K^j \times S \to \mathbb{R}$ representing future utility conditional on savings we define firms’ optimal production and profits are given for each $\hat{p} \in \widehat{P}$ and each $j \in J$ by[13]

$$\tilde{\xi}^j_c(k^j_-, s, \tilde{v}^j, \hat{p}, \hat{a}) = \operatorname{argmax}\{\tilde{u}^j(c^j, s, \hat{p}, \hat{a}) + \bar{\beta}\, \mathrm{e}[\tilde{v}^j(k^j, \hat{a}(s), \cdot)|z] : c^j \in \tilde{b}^j(k^j_-, z)\}.$$

Note that $\tilde{\eta}^j(p, s) = 0$ for all $j \in J$.

2.2 Agents’ features

Consider a model with one good and agents with instantaneous utility function $u^i : C^i \to \mathbb{R}$ concave and increasing. Assume agents have the same discount rate, that is $\beta^i = \bar{\beta}$ for all $i \in I$ and their preferences

[11] Or dividends.

[12] Recall that capital rental firms use physical capital as an input and suppose storage is not available.

[13] Recall that $c^j = (\kappa^j, \ell^j, l^j, m^j, \eta^j, \varsigma^j, k^j)$ for all $c^j \in C^j$.

embodies a type of "volatility" aversion such as the mean-variance preferences, or similarities. More specifically, we assume that the volatility expectation function $\tilde{\nu}^i : E^i \times S \times \Gamma \times \widehat{A} \to \mathbb{R}_+$ represents this behavior.

Define the consumption-asset utility function $\tilde{u}^i : X^i \times S \times \Gamma \times \widehat{A} \to \mathbb{R}_+$ as

$$\tilde{u}^i(x^i, s, \tau, \hat{a}) = u^i(c^i) - \tilde{\nu}^i(\epsilon^i, s, \tau, \hat{a}) \text{ for all } (c^i, \tau, \hat{a}, s) \in C^i \times S \times \Gamma \times \widehat{A}.$$

Allocations of consumption, labor (or leisure) and primary capital endowments are given by the random vector $\hat{e}^i : Z \to \mathbb{R}^7_+$ written as the column vector

$$\hat{e}^i(z) = (\hat{e}^i_\kappa(z), \hat{e}^i_\ell(z), \hat{e}^i_l(z), \hat{e}^i_m(z), \hat{e}^i_\eta(z), \hat{e}^i_\varsigma(z), \hat{e}^i_k(z)) \text{ for all } (i, z) \in I \times Z.$$

Define $\hat{e}(z) = [\hat{e}^i(z) : i \in I] \in \mathbb{R}^{7\times \mathrm{i}}_+$ and recall that $p = [p_\kappa, p_\ell, p_l, p_m, p_\eta, p_\varsigma, p_k]$. We consider as a convention that agents have null capital endowments, that is, $\hat{e}^i_\kappa = 0$ and $\hat{e}^i_k = 0$ for all $i \in I$. The consumers budget correspondence

$$\tilde{b}^i : E^i \times S \times \Gamma \rightrightarrows X^i$$

is given for each $i \in I$ by

$$\tilde{b}^i(\epsilon^i_-, s, \tau) = \{x^i \in X^i : \hat{p}(s)c^i + \hat{q}(s)\epsilon^i \leq (\hat{q}(s) + \hat{\pi}(s))\epsilon^i_- + \hat{p}(s)\hat{e}^i(z)\} \tag{4}$$

for all $(\epsilon^i_-, s, \tau) \in E^i \times S \times \Gamma$. Given a continuous function $\tilde{v}^i : E^i \times S \to \mathbb{R}$ representing future utility conditional on savings then agent i solves the problem

$$\tilde{\xi}^i_x(\epsilon^i_-, s, \tilde{v}^i, \tau, \hat{a}) = \operatorname{argmax}\{\tilde{u}^i(x^i, s, \tau, \hat{a}) + \bar{\beta}\, \mathsf{e}[\tilde{v}^i(\epsilon^i, \hat{a}(s), \cdot)|z] : x^i \in \tilde{b}^i(\epsilon^i_-, s, \tau)\}$$

for each $i \in I$ and each given price transition $\tau \in \Gamma$. This construction will be detailed in Section 4.4.

3 Recursive Equilibrium

Define $\widetilde{X}^i$ as the set of all continuous functions $\tilde{x}^i : E^i \times S \to X^i$, $\widetilde{V}^i$ as the set of all continuous functions $\tilde{v}^i : E^i \times S \to \mathbb{R}$ and $\widehat{V}^i$ as the set of all continuous functions $\hat{v}^i : S \to \mathbb{R}$ for all $i \in I$. Moreover, write $\widetilde{V}^j$ as the set of all continuous functions $\tilde{v}^j : K^j \times S \to \mathbb{R}$ and $\widetilde{C}^j$ as the set of all continuous functions $\tilde{c}^j : K^j \times S \to C^j$ for all $j \in J$. Define $\widehat{V}^j$ analogously.

Definition 3.1. Define for production sectors, the map $\hat{\xi}^j_v : \widetilde{V}^j \times \widehat{P} \times \widehat{A} \to \widetilde{V}^j$ as

$$\tilde{\xi}^j_v(\tilde{v}^j, \hat{p}, \hat{a})(k^j_-, s) = \max\{\tilde{u}^j(c^j, s, \hat{p}, \hat{a}) + \bar{\beta}\, \mathsf{e}[\tilde{v}^j(k^j, \hat{a}(s), \cdot)|z] : c^j \in \tilde{b}^j(k^j_-, z)\} \tag{5}$$

for all $(k^j_-, s) \in K^j \times S$. Moreover, define the map $\hat{\xi}^j_c : \widetilde{V}^j \times \widehat{P} \times \widehat{A} \to \widetilde{C}^j$ for each $j \in J$ as

$$\hat{\xi}^j_c(\tilde{v}^j, \hat{p}, \hat{a})(s) = \tilde{\xi}^j_c(\bar{k}^j_-, s, \tilde{v}^j, \hat{p}, \hat{a}) \text{ for all } s \in S.$$

Define on consumption sector for each $i \in I$ the map $\hat{\xi}^i_v : \widetilde{V}^i \times \Gamma \times \widehat{A} \to \widetilde{V}^i$ for each $s = (\bar{a}_-, z)$

with $\bar{a}_{-} = (\bar{k}_{-}, \bar{\epsilon}_{-})$ as

$$\tilde{\xi}^i_v(\tilde{v}^i, \tau, \hat{a})(\epsilon^i_{-}, s) = \max\left\{\tilde{u}^i(x^i, s, \tau, \hat{a}) + \bar{\beta}\,\mathrm{e}[\tilde{v}^i(\epsilon^i, \hat{a}(s), \cdot)|z] : x^i \in \tilde{b}^i(\epsilon^i_{-}, \hat{a}, s)\right\}$$

and the map $\hat{\xi}^i_x : \widetilde{V}^i \times \Gamma \times \widehat{A} \to \widetilde{X}^i$ as

$$\hat{\xi}^i_x(\tilde{v}^i, \tau, \hat{a})(s) = \tilde{\xi}^i_x(\bar{\epsilon}^i_{-}, s, \tilde{v}^i, \tau, \hat{a}) \text{ for all } s \in S.$$

The definition of recursive equilibrium then follows. First, note that we defined all inputs and outputs with positive amounts. Therefore, to concisely express the market clearing equations, we need to assign negative values to inputs and positive values to outputs. To address this, we introduce a signal functions $\hat{\sigma}_\kappa : \mathbb{R}^7 \to \mathbb{R}^7$, $\hat{\sigma}_k : \mathbb{R}^7_+ \to \mathbb{R}^7$ and $\hat{\sigma}_\varsigma : \mathbb{R}^7 \to \mathbb{R}^7$ as following.

$$\hat{\sigma}_\kappa(y) = (y_\kappa, -y_\ell, -y_l, -y_m, -y_\eta, -y_\varsigma, -y_k) \text{ for all } y \in \mathbb{R}^7_+$$
$$\hat{\sigma}_k(y) = (-y_\kappa, -y_\ell, -y_l, -y_m, -y_\eta, -y_\varsigma, y_k) \text{ for all } y \in \mathbb{R}^7_+$$
$$\hat{\sigma}_\varsigma(y) = (-y_\kappa, -y_\ell, -y_l, -y_m, -y_\eta, y_\varsigma, -y_k) \text{ for all } y \in \mathbb{R}^7_+$$

Definition 3.2. The transition vector $(\hat{c}, \hat{\epsilon}, \tau, \tilde{v}) \in \widehat{C} \times \Gamma \times \widetilde{V}$ is a recursive equilibrium if it satisfies[14] for each $s = (\bar{a}_{-}, z) \in S$

1. $\tilde{v}^i = \tilde{\xi}^i_v(\tilde{v}^i, \tau, \hat{a})$ for all $i \in I$;
2. $(\hat{c}^i, \hat{\epsilon}^i) \in \hat{\xi}^i_x(\tilde{v}^i, \tau, \hat{a})$ for all $i \in I$;
3. $\tilde{v}^j = \hat{\xi}^j_v(\tilde{v}^j, \tau, \hat{a})$ for all $j \in J$;
4. $\hat{c}^j \in \hat{\xi}^j_c(\tilde{v}^j, \tau, \hat{a})$ for all $j \in J$;
5. $\hat{c}^{I+}(s) = \hat{\sigma}_\kappa(\hat{c}^{J_\kappa+}(s)) + \hat{\sigma}_k(\hat{c}^{J_k+}(s)) + \hat{\sigma}_\varsigma(\hat{c}^{J_\varsigma+}(s)) + \hat{e}^{I+}(z)$;
6. $\hat{\epsilon}^{I+}(s) = \bar{\epsilon}^{I+}_{-}$;
7. $\hat{\pi}^j(s) = \tilde{\pi}^j(\hat{c}^j(s), \hat{p}(s))$ for all $j \in J$.

Remark 3.1. Note that all inputs but leisure lead no benefits for consumers and hence optimal choices $\hat{c}^i$ satisfy $(\hat{\kappa}^i, \hat{m}^i, \hat{k}^i) = 0$ for all $i \in I$. Therefore, the Market clearing given in Definition 3.2 Item 5 for each $s = (\bar{a}_{-}, z) \in S$ can be detailed[15] as

1. $\hat{\kappa}^{J_\varsigma+}(s) = \hat{\kappa}^{J_\kappa+}(s) = \bar{k}^{J_\kappa+}_{-}$;
2. $\hat{\ell}^{I+}(s) + \hat{\ell}^{J_\varsigma+}(s) = \hat{e}^{I+}_\ell(z)$;
3. $\hat{l}^{I+}(s) + \hat{l}^{J_k+}(s) = \hat{e}^{I+}_l(z)$;
4. $\hat{m}^{J_k+}(s) = \hat{e}^{I+}_m(z)$;

[14] Recall that $\tau = (\hat{p}, \hat{q}, \hat{\pi})$.
[15] See Remark 4.1 also.

5. $\hat{k}^{J_\kappa+}(s) = \hat{k}^{J_k+}(s) + \gamma \bar{k}_-^{J_\kappa+}$;
6. $\hat{\varsigma}^{I+} = \hat{\varsigma}^{J_\varsigma+}(s) + \hat{e}_\varsigma^{I+}(z)$.

4 The explicit solution

In this section, we will outline all the parameters corresponding to the preferences and technologies described in the recursive equilibrium. Lemma 6.1 in the appendix establishes that explicit boundary conditions are unnecessary, since the first-order conditions governing the optimal allocation endogenously determine boundary values that are themselves optimal.

4.1 Parameters of production sector

Suppose that firms types inside each sector differ in stochastic technical coefficients $\hat{\theta}^j : Z \to \mathbb{R}_+^4$ representing productivity for all $j \in J$ where

$$\hat{\theta}^j(z) = (\hat{\theta}^j_\kappa(z), \hat{\theta}^j_\ell(z), \hat{\theta}^j_l(z), \hat{\theta}^j_m(z)) \text{ for all } z \in Z$$

and scale coefficients

$$\alpha^j = (\alpha^j_\kappa, \alpha^j_\ell, \alpha^j_l, \alpha^j_m) \in \mathbb{R}^4_{++} \text{ for all } j \in J_k \cup J_\kappa.$$

Assumption 4.1. Assume that production sectors are independent, that is,[16]

$$\hat{\theta}^j(z) = \begin{cases} (\hat{\theta}^j_\kappa(z), \hat{\theta}^j_\ell(z), 0, 0) & \text{for } j \in J_\varsigma \\ (0, 0, \hat{\theta}^j_m(z), \hat{\theta}^j_l(z)) & \text{for } j \in J_k \end{cases} \quad \text{for all } z \in Z$$

Capital and good sectors have a technology represented as the following production functions $f^j_k : C^j \times Z \to \mathbb{R}_+$ and $f^j_\varsigma : C^j \times Z \to \mathbb{R}_+$ defined respectively by

$$\begin{aligned} f^j_k(c^j, z) &= \hat{\theta}^j_l(z) \log(1 + l^j/\alpha^j_l) + \hat{\theta}^j_m(z) \log(1 + m^j/\alpha^j_m) \text{ for } j \in J_k \\ f^j_\varsigma(c^j, z) &= \hat{\theta}^j_\kappa(z) \log(1 + \kappa^j/\alpha^j_\kappa) + \hat{\theta}^j_\ell(z) \log(1 + \ell^j/\alpha^j_\ell) \text{ for } j \in J_\varsigma \end{aligned}$$

where $c^j = (\kappa^j, \ell^j, l^j, m^j, \eta^j, \varsigma^j, k^j)$ and $(l^j, m^j) \in L^j \times M^j$ is the amount of primary labor and capital inputs employed on the capital sector and $(\kappa^j, \ell^j) \in K^j \times L^j$ is the amount of capital and labor inputs employed on the consumption goods sector. Note that f^j_k and f^j_ς and the Cobb-Douglas functions have the same isoquants differing only by a linear scale.

Definition 4.1. Define the asset returns of unit portfolio[17] $\hat{r}_\kappa : \widehat{P} \times S \to \mathbb{R}_+$ as

$$\hat{r}_\kappa(\hat{p}, s) = \gamma \hat{p}_k(s) + \hat{p}_\kappa(s) \text{ for all } (\hat{p}, s) \in \widehat{P} \times S$$

[16] Assume to simplify there is no labor employment in capital rental sector.
[17] Or asset payoffs.

Write the second-order moment of $\hat{r}_\kappa$ by $\hat{\mathrm{r}}_\kappa : \widehat{P} \times S \to \mathbb{R}$ as

$$\hat{\mathrm{r}}_\kappa(\hat{p}, s) = (\hat{r}_\kappa(\hat{p}, s))^2 \text{ for all } (\hat{p}, s) \in \widehat{P} \times S$$

We assume that for each firm in capital rental sector the volatility expectation function $\tilde{\nu}^j$ is given for all $j \in J_\kappa$ by

$$\tilde{\nu}^j(k^j, s, \hat{p}, \hat{a}) = \frac{(k^j)^2 \, \mathrm{e}[\hat{\mathrm{r}}_\kappa(\hat{p}, \hat{a}(s), \cdot)|z]}{2\theta^j_\kappa} \text{ for all } (k^j, s) \in K^j \times S. \tag{6}$$

where $\theta^j_\kappa > 0$.

4.2 Parameters of consumption sector

Suppose agents with instantaneous utility function $u^i : C^i \to \mathbb{R}$ defined for each $i \in I$ an each $c^i \in C^i$ by

$$u^i(c^i) = \eta^i + \theta^i_l \log(1 + l^i/\alpha^i_l) + \theta^i_\ell \log(1 + \ell^i/\alpha^i_\ell) + \theta^i_\varsigma \log(1 + \varsigma^i/\alpha^i_\varsigma).$$

where $\theta^i = (\theta^i_\ell, \theta^i_l, \theta^i_\varsigma) \in \mathbb{R}^3_+$ with the scale coefficients

$$\alpha^i = (\alpha^i_\ell, \alpha^i_l, \alpha^i_\varsigma) \in \mathbb{R}^3_{++} \text{ for all } i \in I.$$

Write the non-*numéraire* component $u^i_\eta : L^i \times L^i \times \mathbb{R}_+ \to \mathbb{R}_+$ of u^i as

$$u^i_\eta(\ell^i, l^i, \varsigma^i) = \theta^i_l \log(1 + l^i/\alpha^i_l) + \theta^i_\ell \log(1 + \ell^i/\alpha^i_\ell) + \theta^i_\varsigma \log(1 + \varsigma^i/\alpha^i_\varsigma).$$

Definition 4.2. Define the asset returns of unit portfolio[18] $\hat{r}_\epsilon : \Gamma \times S \to \mathbb{R}^{\mathrm{j}}$ as

$$\hat{r}_\epsilon(\tau, s) = \hat{q}(s) + \hat{\pi}(s) \text{ for all } (\tau, s) \in \Gamma \times S.$$

Write the second-order moment matrix of $\hat{r}$ by $\hat{\mathrm{r}}_\epsilon : \Gamma \times S \to \mathbb{R}^{\mathrm{j} \times \mathrm{j}}$ as

$$\hat{\mathrm{r}}_\epsilon(\tau, s) = \mathrm{tr}(\hat{r}_\epsilon(\tau, s))\hat{r}_\epsilon(\tau, s) \text{ for all } (\tau, s) \in \Gamma \times S$$

Moreover, for each agent $i \in I$ the volatility expectation function is given for each $(\epsilon^i, s) \in E^i \times S$ by

$$\tilde{\nu}^i(\epsilon^i, s, \tau, \hat{a}) = \frac{\mathrm{tr}(\epsilon^i) \, \mathrm{e}[\hat{\mathrm{r}}_\epsilon(\tau, \hat{a}(s), \cdot)|z]\epsilon^i}{2\theta^i_\epsilon} \tag{7}$$

where $\theta^i_\epsilon > 0$.

4.3 Firms' optimal choices

The following proposition exhibits firms optimal choices.

[18] Or asset payoffs.

Proposition 4.1. *Consider the following transition functions $\check{c}^j : S \times \widehat{P} \to C^j$ for $j \in J_k \cup J_c$ and $\check{c}^j : K^j \times S \times \widehat{P} \times \widehat{A} \to C^j$ for $j \in J_\kappa$ defined for each $(s, \hat{p}, \hat{a}) \in S \times \widehat{P} \times \widehat{A}$ by the following*

1. For $j \in J_k$ define $(\check{\kappa}^j, \check{\ell}^j, \check{\eta}^j, \check{\varsigma}^j) = 0$. Moreover,

$$\begin{aligned} \check{l}^j(s,\hat{p}) &= \frac{\hat{\theta}^j_l(z)\hat{p}_k(s)}{\hat{p}_l(s)} - \alpha^j_l \\ \check{m}^j(s,\hat{p}) &= \frac{\hat{\theta}^j_m(z)\hat{p}_k(s)}{\hat{p}_m(s)} - \alpha^j_m \\ \check{k}^j(s,\hat{p}) &= f^j_k(\check{l}^j(s,\hat{p}), \check{m}^j(s,\hat{p})). \end{aligned} \tag{8}$$

2. For $j \in J_\varsigma$ write $(\check{l}^j, \check{m}^j, \check{\eta}^j, \check{k}^j) = 0$. Moreover, define

$$\begin{aligned} \check{\kappa}^j(s,\hat{p}) &= \frac{\hat{\theta}^j_\kappa(z)\hat{p}_\varsigma(s)}{\hat{p}_\kappa(s)} - \alpha^j_\kappa \\ \check{\ell}^j(s,\hat{p}) &= \frac{\hat{\theta}^j_\ell(z)\hat{p}_\varsigma(s)}{\hat{p}_\ell(s)} - \alpha^j_\ell \\ \check{\varsigma}^j(s,\hat{p}) &= f^j_\varsigma\big(\check{\kappa}^j(s,\hat{p}), \check{\ell}^j(s)\big) \end{aligned} \tag{9}$$

3. For $j \in J_\kappa$ write $(\check{\ell}^j, \check{l}^j, \check{m}^j, \check{\eta}^j, \check{\varsigma}^j) = 0$. Moreover, define

$$\check{k}^j(k^j_-, s, \hat{p}, \hat{a}) = \theta^j_\kappa \frac{\bar{\beta}\, \mathrm{e}[\hat{r}_\kappa(\hat{p}, \hat{a}(s), \cdot)|z] - \hat{p}_k(s)}{\mathrm{e}[\hat{\mathrm{r}}_\kappa(\hat{p}, \hat{a}(s), \cdot)|z]} \text{ and } \check{\kappa}^j(k^j_-, s, \hat{p}, \hat{a}) = k^j_- \tag{10}$$

Then for each $j \in J_k \cup J_c$ and each value function $\tilde{v}^j : K^j \times S \to \mathbb{R}$ given by $\tilde{v}^j = 0$ we get

$$\hat{\xi}^j_c(\tilde{v}^j, \hat{p}, \hat{a})(s) = \check{c}^j(s, \hat{p}) \text{ for all } (s, \hat{p}) \in S \times \widehat{P}.$$

Moreover, for each $j \in J_\kappa$ and each value function $\tilde{v}^j : K^j \times S \to \mathbb{R}$ written as

$$\tilde{v}^j(k^j_-, s) = (\gamma \hat{p}_k(s) + \hat{p}_\kappa(s))k^j_- + \hat{v}^j(s) \text{ for all } (k^j_-, s) \in K^j \times S \tag{11}$$

where $\hat{v}^i \in \widehat{V}^i$ is an arbitrary function, we get

$$\hat{\xi}^j_c(\tilde{v}^j, \hat{p}, \hat{a})(s) = \check{c}^j(s, \hat{p}, \hat{a}) \text{ for all } (s, \hat{p}, \hat{a}) \in S \times \widehat{P} \times \widehat{A}$$

Proof: See Raad et al. (2025). Observe that, since all objective functions are concave, solving the first-order conditions is sufficient to characterize the global optimum. For further details, refer to Theorem 6.1 in the appendix. We explicitly derive the solutions to the first-order conditions. □

4.4 Agents' optimal choices

The agents' demand are given by the following result. First consider the definition of Bellman equation.

Proposition 4.2. *Consider a transition vector $\check{\epsilon}^i : S \times \Gamma \times \widehat{A} \to E^i$ satisfying for each $i \in I$ and all $(s, \tau) \in S \times \Gamma$ by*

$$\mathrm{e}[\hat{\mathrm{r}}_\epsilon(\tau, \hat{a}(s), \cdot)|z]\check{\epsilon}^i(s, \tau, \hat{a}) = \mathrm{tr}\left(\bar{\beta}\, \mathrm{e}[\hat{r}_\epsilon(\tau, \hat{a}(s), \cdot)|z] - \hat{q}(s)\right)\theta^i_\epsilon. \tag{12}$$

Define the following transition functions $\check{c}^i : E^i \times S \times \Gamma \to C^i$ for $i \in I$ and each $(\epsilon^i_-, s, \tau) \in E^i \times S \times \Gamma$

$$\begin{aligned}
\check{\kappa}^i_-(\epsilon^i_-, s, \tau) = 0 \quad \check{\ell}^i(\epsilon^i_-, s, \tau) &= \frac{\theta^i_\ell}{\hat{p}_\ell(s)} - \alpha^i_\ell \\
\check{l}^i(\epsilon^i_-, s, \tau) = \frac{\theta^i_l}{\hat{p}_l(s)} - \alpha^i_l \quad \check{m}^i(\epsilon^i_-, s, \tau) &= 0 \\
\check{k}^i(\epsilon^i_-, s, \tau) = 0 \quad \check{\varsigma}^i(\epsilon^i_-, s, \tau) &= \frac{\theta^i_\varsigma}{\hat{p}_\varsigma(s)} - \alpha^i_\varsigma \\
\check{\eta}^i(\epsilon^i_-, s, \tau) = \hat{r}_\epsilon(s, \tau)\epsilon^i_- + \hat{p}(s)\hat{e}^i(z) &- \hat{q}(s)\check{\epsilon}^i(s, \tau, \hat{a}) \\
&- \hat{p}_\ell(s)\check{\ell}^i(\epsilon^i_-, s, \tau) - \hat{p}_l(s)\check{l}^i(\epsilon^i_-, s, \tau) - \hat{p}_\varsigma(s)\check{\varsigma}^i(\epsilon^i_-, s, \tau)
\end{aligned} \tag{13}$$

and write $\check{x}^i : E^i \times S \to X^i$ by $\check{x}^i = (\check{c}^i, \check{\epsilon}^i)$.

Then for each value function $\tilde{v}^i \in \widetilde{V}^i$ for $i \in I$ written as

$$\tilde{v}^i(\epsilon^i_-, s) = (\hat{q}(s) + \hat{\pi}(s))\epsilon^i_- + \hat{v}^i(s) \text{ for all } s \in S \tag{14}$$

where $\hat{v}^i \in \widehat{V}^i$ is an arbitrary function, we get

$$\check{x}^i(\bar{\epsilon}^i_-, s, \tau) \in \hat{\xi}^i_x(\tilde{v}^i, \tau, \hat{a})(s) \text{ for each } (\bar{\epsilon}^i_-, s, \tau) \in E^i \times S \times \Gamma.$$

Proof: See Raad et al. (2025). □

Remark 4.1. Market clearing conditions given in Definition 3.2 can be clarified by the Walras Law. Indeed, suppose that $\hat{\pi}^{J+}(s) = \sum_{j \in J} \tilde{\pi}^j(\hat{c}^j(s), \hat{p}(s))$. Adding up the budget constrains given in (4) we get[19]

$$\hat{p}(s)(\hat{c}^{I+}(s) - \hat{e}^{I+}(z)) + \hat{q}(s)(\hat{\epsilon}^{I+} - \epsilon^{I+}_-) = \hat{\pi}^{J+}(s)$$

and hence by (1), (2), and (3) we get

$$\begin{aligned}
0 = \hat{p}_\kappa(s)(\bar{k}^{J_\kappa+}_-(s) - \hat{\kappa}^{J_\varsigma+}(s)) + \hat{p}_\ell(s)(\hat{\ell}^{I+}(s) + \hat{\ell}^{J_\varsigma+}(s) - \hat{e}^{I+}_\ell(z)) \\
+ \hat{p}_l(s)(\hat{l}^{I+}(s) + \hat{l}^{J_k+}(s) - \hat{e}^{I+}_l(z)) + \hat{p}_m(s)(\hat{m}^{J_k+}(s) - \hat{e}^{I+}_m(z)) \\
+ \hat{p}_\eta(s)(\hat{\eta}^{I+}(s) - \hat{e}^{I+}_\eta(z)) + \hat{p}_\varsigma(s)(\hat{\varsigma}^{I+}(s) - \hat{\varsigma}^{J_\varsigma+}(s) - \hat{e}^{I+}_\varsigma(z)) \\
+ \hat{p}_k(s)\left(\hat{k}^{J_\kappa+}(s) - \hat{k}^{J_k+}(s) - \gamma\bar{k}^{J_\kappa+}_-\right) + \hat{q}(s)(\hat{\epsilon}^{I+}(s) - \bar{\epsilon}^{I+}_-).
\end{aligned}$$

[19]Recall that $\epsilon^{I+}_- = 1 \in \mathbb{R}^{\mathrm{j}}_+$.

4.5 Recursive equilibrium

In this section we will show the recursive relations relying equilibrium prices, based on previous sections optimal choices.

Remark 4.2. Note that the optimal asset choices $\tilde{a} = (\tilde{k}, \tilde{\epsilon})$, as given in (10) and (12), do not depend on the previous portfolio $a_{-} = (k_{-}, \epsilon_{-})$. This implies that the variables outside equilibrium do not alter the asset transition function. Thus, we can select the aggregate capital variable as the only relevant one within the state space. We then choose any asset transition that maintains the same level of aggregate capital and equities. This leads to the following definition.

Definition 4.3. Write $\bar{\bar{\epsilon}}^i_{-} = (1, \cdots, 1)/\mathrm{i} \in \mathbb{R}^{\mathrm{j}}_{+}$ and $\bar{\bar{\epsilon}}_{-} = [\bar{\bar{\epsilon}}^i_{-}]_{i \in I}$. Define the minimal state space as $S_{\mathrm{m}} = A_{\mathrm{m}} \times Z$ where

$$A_{\mathrm{m}} = \{\big((k^*_{-}, \cdots, k^*_{-})/\mathrm{j}_{\kappa}, \bar{\bar{\epsilon}}_{-}\big) : k^*_{-} \in K^{J_{\kappa^+}}\}.$$

The next assumption implies that the minimal state space can be identified[20] as the set $K^{J_{\kappa^+}} \times Z$ with a typical element defined as $s = (k^{J_{\kappa^+}}_{-}, z) \in \mathbb{R}_{+} \times Z$. Moreover, $S_{\mathrm{m}} \subset S$ since K is convex.

Assumption 4.2. Suppose that agents face no disutility from labor in capital-producing sectors. Specifically, assume that $\theta^i_l = 0$ for all $i \in I$.

Remark 4.3. In this model, the aggregate capital transition plays the role of a sufficient statistic to characterize the equilibrium, as it is specified exogenously. We shall therefore define both the aggregate capital transition function and the complete asset transition mapping in order to construct the operator whose fixed point characterizes the recursive equilibrium. More precisely, since $\check{l}^i(\epsilon^i_{-}, s, \tau) = 0$ for all $(\epsilon^i_{-}, s, \tau) \in E^i \times S \times \Gamma$ and all $i \in I$ then equilibrium prices must satisfy:

$$\hat{e}^{I+}_l(z) = \check{l}^{J_{k^+}}(s, \hat{p}) = \frac{\hat{\theta}^{J_{k^+}}_l(z)\hat{p}_k(s)}{\hat{p}_l(s)} - \alpha^{J_{k^+}}_l$$

$$\hat{e}^{I+}_m(z) = \check{m}^{J_{k^+}}(s, \hat{p}) = \frac{\hat{\theta}^{J_{k^+}}_m(z)\hat{p}_k(s)}{\hat{p}_m(s)} - \alpha^{J_{k^+}}_m.$$

Therefore,

$$\frac{\hat{p}_k(s)}{\hat{p}_l(s)} = \frac{\hat{e}^{I+}_l(z) + \alpha^{J_{k^+}}_l}{\hat{\theta}^{J_{k^+}}_l(z)} \text{ and } \frac{\hat{p}_k(s)}{\hat{p}_m(s)} = \frac{\hat{e}^{I+}_m(z) + \alpha^{J_{k^+}}_m}{\hat{\theta}^{J_{k^+}}_m(z)}$$

and hence

$$\check{l}^j(s, \hat{p}) = \hat{\theta}^j_l(z)\frac{\hat{e}^{I+}_l(z) + \alpha^{J_{k^+}}_l}{\hat{\theta}^{J_{k^+}}_l(z)} - \alpha^j_l$$

$$\check{m}^j(s, \hat{p}) = \hat{\theta}^j_m(z)\frac{\hat{e}^{I+}_m(z) + \alpha^{J_{k^+}}_m}{\hat{\theta}^{J_{k^+}}_m(z)} - \alpha^j_m$$

$$\check{k}^j(s, \hat{p}) = f^j_k(\check{l}^j(s, \hat{p}), \check{m}^j(s, \hat{p})).$$

[20] This space can be interpreted as the set of asset allocations that are equally distributed between firms and investors.

This implies that $\check{k}^j$ depends only on z for all $j \in J_k$. Thus we have the following definition by writing $\hat{k}^j(z) = \check{k}^j(\bar{a}_{-}, z, \hat{p})$ for all $j \in J_k$ and all $z \in Z$.

Definition 4.4. Define $\check{k}_{\mathrm{m}} : S_{\mathrm{m}} \to K^{J_{\kappa^+}}$ by

$$\check{k}_{\mathrm{m}}(s) = \gamma k_{-}^{J_{\kappa^+}} + \hat{k}^{J_{\kappa^+}}(z) \text{ for all } s \in S$$

and $\check{a}_{\mathrm{m}} : S_{\mathrm{m}} \to A_{\mathrm{m}}$

$$\check{a}_{\mathrm{m}}(s) = \big((1, \cdots, 1)\check{k}_{\mathrm{m}}(s)/\mathrm{j}_{\kappa}, \bar{\bar{\epsilon}}_{-}\big) \text{ for all } s \in S$$

Remark 4.4. Since all price transition functions are constant outside of S_{m}, these functions evaluated at $\hat{a}(s)$ will yield the same value as when evaluated at $\hat{a}_{\mathrm{m}}(s)$ for any $s \in S$. In fact, the function $\hat{a}_{\mathrm{m}}(s)$ represents an egalitarian redistribution of assets between agents and firms in the subsequent period. We now construct an operator whose fixed point will be the candidate for equilibrium. This operator will also be constant outside of S_{m}. This is the main reason why the space S_{m} is minimal.

The recursive equilibrium existence is characterized by the following Theorem. First define $\check{\Gamma} = \check{P} \times \check{Q}$ and $\check{V}$ defined over S_{m}. Moreover, define $\mathbb{R}^{\mathrm{j}}_{\mathrm{m}}$ as the space of all continuous functions $\check{\pi} : S_{\mathrm{m}} \to \mathbb{R}^{\mathrm{j}}_{\mathrm{m}}$.

Definition 4.5. Define the following auxiliary functions

1. Profits $\check{\pi}^j : \check{P} \to \mathbb{R}^{\mathrm{j}}_{\mathrm{m}}$ defined for each $j \in J$ by

$$\check{\pi}^j(\check{p})(s) = \begin{cases} \tilde{\pi}^j(\check{c}^j(s, \check{p}), \check{p}) \text{ if } j \in J_k \cup J_\varsigma \\ \tilde{\pi}^j(\check{c}^j(s, \check{p}, \check{a}_{\mathrm{m}}), \check{p}) \text{ if } j \in J_\kappa. \end{cases}$$

2. Capital stochastic discount factor $\check{\delta}_\kappa : \mathbb{R}_+ \times \check{\Gamma} \times S_{\mathrm{m}} \to \mathbb{R}$ defined by

$$\check{\delta}_\kappa(y, \check{\tau}, s) = \bar{\beta} - \hat{r}_\kappa(\check{p}, s)y$$

for all $(y, \tau, s) \in \mathbb{R}_+ \times \check{\Gamma} \times S_{\mathrm{m}}$

3. Equities stochastic discount factor $\check{\delta}_\epsilon : \mathbb{R}^{\mathrm{j}} \times \check{\Gamma} \times S_{\mathrm{m}} \to \mathbb{R}_+$ by

$$\check{\delta}_\epsilon(y, \check{\tau}, s) = \bar{\beta} - \hat{r}_\epsilon(\check{p}, \check{q}, \check{\pi}(\check{p}), s)y \text{ for all } (y, \check{\tau}, s) \in \mathbb{R}^{\mathrm{j}} \times \check{\Gamma} \times S$$

4. Partial utility function[21] $\check{u}^i : \Gamma \times S_{\mathrm{m}} \to \mathbb{R}_+$ defined by

$$\begin{aligned} \check{u}^i(\tau, s) = {} & \hat{p}(s)\hat{e}^i(z) - \hat{q}(s)\check{\epsilon}^i(s, \tau, \hat{a}) \\ & - \hat{p}_\ell(s)\check{\ell}^i(\bar{\epsilon}^i_{-}, s, \tau) - \hat{p}_l(s)\check{l}^i(\bar{\epsilon}^i_{-}, s, \tau) - \hat{p}_\varsigma(s)\check{\varsigma}^i(\bar{\epsilon}^i_{-}, s, \tau) \\ & + u^i_\eta(\check{\ell}^i(\bar{\epsilon}^i_{-}, s, \tau), \check{l}^i(\bar{\epsilon}^i_{-}, s, \tau), \check{\varsigma}^i(\bar{\epsilon}^i_{-}, s, \tau)) \end{aligned}$$

[21] The partial utility function does not include the *numéraire* component corresponding to the payoffs of equities.

Definition 4.6. Define the operator $\check{\xi} : \check{\Gamma} \to \widehat{\mathbb{R}}^7_+ \times \widehat{\mathbb{R}}^{\mathrm{j}}_+$ as following. Let $\check{\tau} = (\check{p}, \check{q}) \in \widehat{\Gamma}$ be an arbitrary price transition. Write $\check{\xi} = (\check{\xi}_p, \check{\xi}_q)$ where[22] for each $s \in S_{\mathrm{m}}$

$$
\begin{aligned}
\check{\xi}_{p\kappa}(\check{\tau})(s) &= \hat{\theta}^{J_\varsigma +}_{\kappa}(z)\check{p}_\varsigma(s)(\bar{k}^{J_\kappa +}_{-} + \alpha^{J_\varsigma +}_{\kappa})^{-1} \\
\check{\xi}_{p\ell}(\check{\tau})(s) &= (\hat{\theta}^{J_\varsigma +}_{\ell}(z)\check{p}_\varsigma(s) + \theta^{I+}_{\ell})(\hat{e}^{I+}_{\ell}(z) + \alpha^{I+}_{\ell} + \alpha^{J_\varsigma +}_{\ell})^{-1} \\
\check{\xi}_{pl}(\check{\tau})(s) &= \hat{\theta}^{J_k +}_{l}(z)\check{p}_k(s)(\hat{e}^{I+}_{l}(z) + \alpha^{J_k +}_{l})^{-1} \\
\check{\xi}_{pm}(\check{\tau})(s) &= \hat{\theta}^{J_k +}_{m}(z)\check{p}_k(s)(\hat{e}^{I+}_{m}(z) + \alpha^{J_k +}_{m})^{-1} \\
\check{\xi}_{p\eta}(\check{\tau})(s) &= 1 \\
\check{\xi}_{p\varsigma}(\check{\tau})(s) &= \theta^{I+}_{\varsigma}(\check{\varsigma}^{J_\varsigma +}(s, \check{p}) + \hat{e}^{I+}_{\varsigma}(z) + \alpha^{I+}_{\varsigma})^{-1} \\
\check{\xi}_{pk}(\check{\tau})(s) &= \mathrm{e}[\check{\delta}_\kappa(\check{k}_{\mathrm{m}}(s)/\theta^{J_\kappa +}_{\kappa}, \check{\tau}, \check{a}_{\mathrm{m}}(s), \cdot\,)\hat{r}_\kappa(\check{p}, \check{a}_{\mathrm{m}}(s), \cdot\,)|z] \\
\check{\xi}_{q}(\check{\tau})(s) &= \mathrm{e}[\check{\delta}_\epsilon\big(\bar{\epsilon}^{I+}_{-}/\theta^{I+}_{\epsilon}, \check{\tau}, \check{a}_{\mathrm{m}}(s), \cdot\,\big)\hat{r}_\epsilon(\check{p}, \check{q}, \check{\pi}(\check{p}), \check{a}_{\mathrm{m}}(s), \cdot\,)|z]
\end{aligned}
\tag{15}
$$

Definition 4.7. Given an arbitrary vector function $\tau \in \Gamma$, define the operator of value functions $\check{\xi}_\tau : \check{V} \to \check{V}$ for each $s \in S_{\mathrm{m}}$ as

$$
\begin{aligned}
\check{\xi}^j_\tau(\check{v})(s) &= 0 \text{ for } j \in J_k \cup J_\varsigma \\
\check{\xi}^j_\tau(\check{v})(s) &= \mathrm{e}[\check{\delta}_\kappa((2\theta^j_\kappa)^{-1}\check{k}^j(\bar{k}^j_{-}, s, \check{p}, \check{a}_{\mathrm{m}}), \check{a}_{\mathrm{m}}(s), \cdot\,)\hat{r}_\kappa(\check{p}, \check{a}_{\mathrm{m}}(s), \cdot\,)|z] \\
&\quad \cdot\, \check{k}^j(\bar{k}^j_{-}, s, \check{p}, \check{a}_{\mathrm{m}}) - \check{p}_k(s)\check{k}^j(\bar{k}^j_{-}, s, \check{p}, \check{a}) \\
&\quad + \bar{\beta}\,\mathrm{e}[\check{v}^j(\check{a}_{\mathrm{m}}(s), \cdot\,)|z] \text{ for } j \in J_\kappa \\
\check{\xi}^i_\tau(\check{v})(s) &= \mathrm{e}[\check{\delta}_\epsilon((2\theta^i_\epsilon)^{-1}\check{\epsilon}^i(s, \tau, \check{a}_{\mathrm{m}}), \tau, \check{a}_{\mathrm{m}}(s), \cdot\,)\hat{r}_\epsilon(\tau, \check{a}_{\mathrm{m}}(s), \cdot\,)|z]\check{\epsilon}^i(s, \tau, \check{a}_{\mathrm{m}}) \\
&\quad + \check{u}^i(\tau, s) + \bar{\beta}\,\mathrm{e}[\check{v}^i(\check{a}_{\mathrm{m}}(s), \cdot\,)|z] \text{ for } i \in I
\end{aligned}
\tag{16}
$$

The following definition constitutes the first step toward transforming the transition functions, obtained as the fixed point of the operator $\check{\xi}$ and defined on S_{m}, into a recursive equilibrium defined on S.

Definition 4.8. Consider $\check{\tau} = (\check{p}, \check{q}, \check{v}) \in \check{\Gamma} \times \check{V}$. Define the aggregator function[23] $\hat{s}_{\mathrm{m}} : S \to S_{\mathrm{m}}$ by

$$
\hat{s}_{\mathrm{m}}(s) = \big((k^{J_\kappa +}_{-}, \cdots, k^{J_\kappa +}_{-})/\mathrm{j}_\kappa, \bar{\bar{\epsilon}}_{-}, z\big) \text{ for all } s \in S
$$

Define $\hat{\tau} = (\hat{p}, \hat{q}, \hat{\pi}, \hat{v}) \in \Gamma \times \check{V}$ by

$$
\hat{p}(s) = \check{p}(\hat{s}_{\mathrm{m}}(s)) \quad \hat{q}(s) = \check{q}(\hat{s}_{\mathrm{m}}(s)) \quad \hat{\pi}(s) = \check{\pi}(\check{p})(\hat{s}_{\mathrm{m}}(s)) \text{ for all } s \in S.
$$

[22] Recall that $\check{\xi}_p = (\check{\xi}_{p\kappa}, \check{\xi}_{p\ell}, \check{\xi}_{pl}, \check{\xi}_{pm}, \check{\xi}_{p\eta}, \check{\xi}_{p\varsigma}, \check{\xi}_{pk})$.

[23] Which consists in a change of variable that leads to equally distribution.

Define $\hat{k} \in \widehat{K}$, $\hat{\epsilon} \in \widehat{E}$, $\hat{a} \in \widehat{A}$ and $\tilde{v} \in \widetilde{V}$ by

$$\hat{k}^j_{\mathrm{m}}(s) = \check{k}^j(\bar{k}^j_{\text{-}}, \hat{s}_{\mathrm{m}}(s), \check{p}, \check{a}_{\mathrm{m}}) \text{ for all } s \in S \text{ and } j \in J_\kappa$$
$$\hat{\epsilon}^i_{\mathrm{m}}(s) = \check{\epsilon}^i(\hat{s}_{\mathrm{m}}(s), \check{p}, \check{q}, \check{\pi}(\check{p}), \check{a}_{\mathrm{m}}) \text{ for all } s \in S \text{ and } i \in I$$
$$\hat{a}_{\mathrm{m}}(s) = (\hat{k}_{\mathrm{m}}(s), \hat{\epsilon}_{\mathrm{m}}(s)) \text{ for all } s \in S$$
$$\tilde{v}^j_{\mathrm{m}}(k^j_{\text{-}}, s) = (\gamma\hat{p}_k(s) + \hat{p}_\kappa(s))k^j_{\text{-}} + \check{v}^j(\hat{s}_{\mathrm{m}}(s)) \text{ for all } (k^j_{\text{-}}, s) \in K^j \times S$$
$$\tilde{v}^i_{\mathrm{m}}(\epsilon^i_{\text{-}}, s) = (\hat{q}(s) + \hat{\pi}(s))\epsilon^i_{\text{-}} + \check{v}^i(\hat{s}_{\mathrm{m}}(s)) \text{ for all } (\epsilon^i_{\text{-}}, s) \in E^i \times S$$

Moreover, define all remaining optimal choices on S analogously by

$$(\hat{c}^i, \hat{\epsilon}^i) = \hat{\xi}^i_x(\tilde{v}^i_{\mathrm{m}}, \hat{p}, \hat{q}, \hat{\pi}, \hat{a}_{\mathrm{m}}) \text{ for all } i \in I$$
$$\hat{c}^j = \hat{\xi}^j_c(\tilde{v}^j_{\mathrm{m}}, \hat{p}, \hat{q}, \hat{\pi}, \hat{a}_{\mathrm{m}}) \text{ for all } j \in J$$
$$\tilde{v}^j = \tilde{\xi}^j_v(\tilde{v}^j_{\mathrm{m}}, \hat{p}, \hat{a}_{\mathrm{m}}) \text{ for all } j \in J$$
$$\tilde{v}^i = \tilde{\xi}^i_v(\tilde{v}^i_{\mathrm{m}}, \hat{\tau}, \hat{a}_{\mathrm{m}}) \text{ for all } i \in I$$

The next proposition is crucial for understanding *how the state space can be reduced*, which in this context will depend on the specific features of the model. Note that in cases where preferences are not quasi-linear, this argument no longer holds, as prices will then depend on the distribution of assets.

Proposition 4.3. *Suppose that $\check{\xi}$ has a fixed point $\check{\tau} \in \Gamma$ and that $\check{g} : S_{\mathrm{m}} \times \check{Y} \to Y$ is a transition function where $\check{Y}$ is the space of all continuous functions $\check{y} : S_{\mathrm{m}} \to Y$. Given $\check{y} \in \check{Y}$ define $\hat{g} : S \to Y$ by $\hat{g}(s) = \check{g}(\hat{s}_{\mathrm{m}}(s))$ for all $s \in S$. Consider $\check{a}_{\mathrm{m}}$ given in Definition 4.8. Then*

$$\hat{g}(\hat{a}_{\mathrm{m}}(s), z') = \check{g}(\check{a}_{\mathrm{m}}(\hat{s}_{\mathrm{m}}(s)), z') \textit{ for all } (s, z') \in S \times Z.$$

Proof: See Raad et al. (2025). □

The operator $\check{\xi}$ is constructed to obtain the market clearing conditions at its fixed point. Indeed, by the next proposition clarifies this claim.

Proposition 4.4. *Suppose that $\check{\xi}$ has a fixed point $\check{\tau} \in \Gamma$ and that $\check{v}$ is a fixed point of $\check{\xi}_{\check{\tau}}$. Consider the transition vector function $(\hat{c}^I, \hat{\epsilon}^I, \hat{c}^J)$ and $\tilde{v}$ given by Definition 4.8. Then for each $s \in S$*

$$\hat{k}^{J_\kappa}(s) = \hat{k}^{J_\kappa}_{\mathrm{m}}(s) \quad \hat{\epsilon}^I(s) = \hat{\epsilon}^I_{\mathrm{m}}(s) \quad \hat{\epsilon}^{I+}(s) = \bar{\epsilon}^{I+}_{\text{-}}$$
$$\hat{c}^{I+}(s) = \hat{\sigma}_\kappa(c^{J_\kappa+}(s)) + \hat{\sigma}_k(c^{J_k+}(s)) + \hat{\sigma}_\varsigma(c^{J_\varsigma+}(s)) + \hat{e}^{I+}(z)$$
$$\hat{a}(s) = \hat{a}_{\mathrm{m}}(s) \quad \tilde{v}(s) = \tilde{v}_{\mathrm{m}}(s)$$

Proof: See Raad et al. (2025). □

Proposition 4.5. *Consider the operators $\check{\xi}$ and $\check{\xi}_\tau$ as in Definition 4.6. Under regularity Assumptions we can state that*

1. *$\check{\xi}(\Gamma) \subset \Gamma$ and $\check{\xi}_\tau(\check{V}) \subset \check{V}$ for all $\tau \in \Gamma$*

2. *$\check{\xi}$ and $\check{\xi}_\tau$ for each fixed $\tau \in \Gamma$ are contractions and hence have a unique fixed point.*

Proof: See Raad et al. (2025). □

Theorem 4.1. *Suppose that Assumption* **??** *holds. Then $(\hat{c}, \hat{a}, \hat{p}, \hat{q}, \tilde{v})$ given as in Definition 4.8 is a recursive equilibrium.*

Proof: See Raad et al. (2025). □

4.6 Example

In order to illustrate the structure and numerical implementation of a general recursive equilibrium with heterogeneous agents, we consider a calibrated example that specifies the key parameter values and functional forms governing the economy. The model features heterogeneous firms indexed by technological types and sectoral roles, operating under uncertain productivity that vary across states of nature. Preferences and technology parameters are disaggregated into aggregate and agent-specific components, with scaling adjustments to capture firm-level heterogeneity in factor intensities and productivity. The endowment processes and production coefficients are specified as functions of exogenous state variables, allowing for the computation of stationary allocations and price mappings across states. The set of parameters below characterizes one such numerical instance used in the computation of a recursive competitive equilibrium. The sequential equilibrium is implemented by recursive equilibrium as in Raad and Woźny (2019). In what follows, we present the values used in this example, followed by graphical representations of key functions and equilibrium objects derived from them

First, we consider the characterization of exogenous uncertainty and main parameters.

$$Z = \{0, 1, 2\}, \quad \gamma = 1/2, \quad \beta = 1/2,$$
$$\mathrm{j}_k = 200, \quad \mathrm{j}_\varsigma = 200, \quad \mathrm{j}_\kappa = 500, \quad \mathrm{j} = \mathrm{j}_k + \mathrm{j}_\varsigma + \mathrm{j}_\kappa$$

The exogenous shocks are represented by the function sh, which captures independent and identically distributed (i.i.d.) equiprobable events over the finite state space $Z = 0, 1, 2$. The amplitude of these shocks is governed by the parameter $\phi = 0.1$, which specifies their relative magnitude. Each state $z \in Z$ corresponds to a sector-specific positive shock of 10%: $z = 0$ is associated with the capital rental sector, $z = 1$ with the capital production sector, and $z = 2$ with the consumption goods sector. In each state, the positively affected sector experiences a $+10\%$ deviation, while the remaining two sectors experience a symmetric -10% deviation. This construction guarantees that the aggregate effect of shocks is neutral, thereby ensuring market completeness across states.

State probabilities and shocks:

$$\mathrm{sh}_0(z) = (1+\phi)(1-z)(2-z)/2 + (1-\phi)z(2-z) + (1-\phi)(z-1)z/2,$$
$$\mathrm{sh}_1(z) = (1-\phi)(1-z)(2-z)/2 + (1+\phi)z(2-z) + (1-\phi)(z-1)z/2,$$
$$\mathrm{sh}_2(z) = (1-\phi)(1-z)(2-z)/2 + (1-\phi)z(2-z) + (1+\phi)(z-1)z/2$$

Endowment scaling parameters:

$$\epsilon_{\text{-}}^{k} = \mathrm{j}_k, \quad \epsilon_{\text{-}}^{\varsigma} = \mathrm{j}_\varsigma, \quad \epsilon_{\text{-}}^{\kappa} = \mathrm{j}_\kappa$$

1. **Aggregate endowments**

$$\begin{aligned} e_\ell^{I+}(z) &= 2\mathrm{sh}_1(z), \\ e_m^{I+}(z) &= 2\mathrm{sh}_1(z), \\ e_\ell^{j}(z) &= 2\mathrm{sh}_2(z), \\ e_\varsigma^{I+}(z) &= 2\mathrm{sh}_2(z) \end{aligned}$$

2. **Aggregate parameters**

$$\begin{aligned} \theta_\ell^{J_k+}(z) &= \mathrm{sh}_1(z), \\ \theta_m^{J_k+}(z) &= \mathrm{sh}_1(z), \\ \theta_\kappa^{J_\varsigma+}(z) &= \mathrm{sh}_2(z), \\ \theta_\ell^{J_\varsigma+}(z) &= \mathrm{sh}_2(z), \\ \theta_\kappa^{J_\kappa+} &= 2000, \\ \theta_\ell^{I+} &= 0, \\ \theta_\ell^{I+} &= 200, \\ \theta_\varsigma^{I+} &= 1000, \\ \theta_\epsilon^{I+} &= 10000 \end{aligned}$$

3. **Agent-specific parameters**

$$\begin{aligned} \theta_\ell^{j}(z) &= \frac{\theta_\ell^{J_k+}(z)}{\mathrm{j}_k}, \\ \theta_m^{j}(z) &= \frac{\theta_m^{J_k+}(z)}{\mathrm{j}_k}, \\ \theta_\kappa^{j}(z) &= \frac{\theta_\kappa^{J_\varsigma+}(z)}{\mathrm{j}_\varsigma}, \\ \theta_\ell^{j}(z) &= \frac{\theta_\ell^{J_\varsigma+}(z)}{\mathrm{j}_\varsigma}, \\ \theta_\kappa^{j_\kappa} &= \frac{\theta_\kappa^{J_\kappa+}}{\mathrm{j}_\kappa} \end{aligned}$$

4. **Scale aggregate parameters**

$$
\begin{aligned}
\text{scale} &= 2 \\
\alpha_\ell^{J_k{}^+} &= \text{scale}, \\
\alpha_m^{J_k{}^+} &= \text{scale}, \\
\alpha_\kappa^{J_\varsigma{}^+} &= \text{scale}, \\
\alpha_\ell^{J_\varsigma{}^+} &= \text{scale}, \\
\alpha_\varsigma^{I^+} &= 10, \\
\alpha_\ell^{I^+} &= 0, \\
\alpha_\ell^{I^+} &= 10
\end{aligned}
$$

5. **Firm-specific scale parameters**

$$
\begin{aligned}
\alpha_\ell^j &= \frac{\alpha_\ell^{J_k{}^+}}{\mathrm{j}_k}, \\
\alpha_m^j &= \frac{\alpha_m^{J_k{}^+}}{\mathrm{j}_k}, \\
\alpha_\kappa^j &= \frac{\alpha_\kappa^{J_\varsigma{}^+}}{\mathrm{j}_\varsigma}, \\
\alpha_\ell^j &= \frac{\alpha_\ell^{J_\varsigma{}^+}}{\mathrm{j}_\varsigma}
\end{aligned}
$$

The figure below presents the recursive capital transition function, which maps the current aggregate capital and the exogenous shock realization into next period's aggregate capital. This function is central to understanding the endogenous dynamics of capital accumulation in the model. It encapsulates the equilibrium responses of heterogeneous agents to current prices and states, aggregating individual decisions into a law of motion for the aggregate capital stock. The graph reveals the presence of two fixed points—values of capital for which the economy remains stationary under a given shock—which demarcate distinct dynamic regimes. One corresponds to a low-capital steady state, while the other reflects a high-capital equilibrium path. These fixed points help characterize the cyclical nature of transitions, as the economy endogenously fluctuates within their basin of attraction, even in the absence of frictions or policy shocks.

The first three panels below illustrate the cyclical behavior of efficient aggregate capital allocations arising from endogenous trade flows among heterogeneous agents. These capital cycles are not imposed externally but emerge naturally from the decentralized equilibrium. Each agent's response aggregate exogenous shocks, combined with sectoral specialization and financial incentives, gives rise to aggregate fluctuations in the supply and use of capital across production and rental activities. The observed cycles are *persistent* and reflect a purely market-driven adjustment mechanism in which the economy reallocates capital resources dynamically in response to underlying productivity and preference shocks. This highlights the internal propagation capacity of the model and underscores the role of heterogeneity in

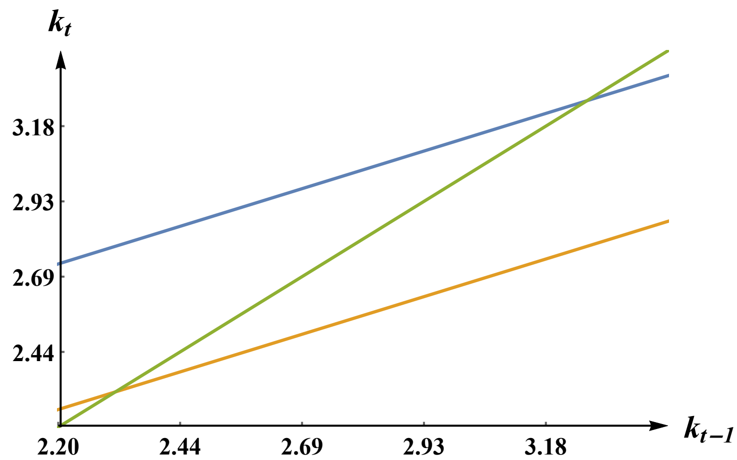
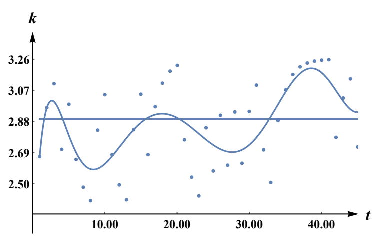
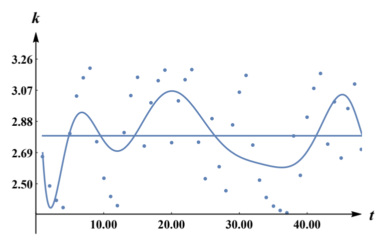


Figure 1: Recursive capital transition function with shock-dependent dynamics and fixed points.

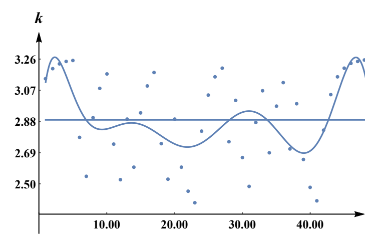
generating smooth yet nontrivial endogenous capital flows over time.



(a) Scenario 1



(b) Scenario 2



(c) Scenario 3

Figure 2: Cyclic behavior of aggregate capital across three I.I.d simulations.

Complementing this analysis, the fourth figure presents the equilibrium path of capital prices, which exhibit their own cyclical dynamics. These prices act as counter-cyclical market forces: when aggregate capital supply expands due to favorable conditions in one sector, prices adjust downward to incentivize reallocation toward other uses, and vice versa. Such price responses provide an essential balancing mechanism between the demand and supply of capital across rental and productive applications. This endogenous price adjustment process reinforces market completeness and ensures that the allocation of capital remains efficient despite the presence of shocks and agent heterogeneity.

5 Conclusion

This paper shows that persistent real business cycles can emerge endogenously from decentralized trade flows among heterogeneous agents in an economy with complete markets and no frictions. In contrast to the dominant view in the business cycle literature, which relies on market imperfections to justify aggregate fluctuations, our model demonstrates that cyclical dynamics are not only consistent with full efficiency, but are in fact a natural consequence of decentralized coordination. When all markets clear and agents act optimally under complete information, intertemporal and cross-sectional reallocations of capital and consumption can still generate nontrivial aggregate dynamics.

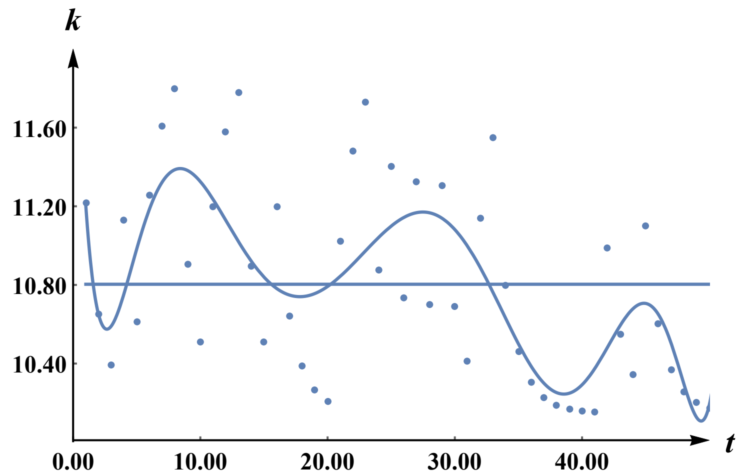


Figure 3: Cyclic behavior of capital prices as an endogenous adjustment force.

The core mechanism relies on a recursive general equilibrium framework with a minimal state space and globally heterogeneous agents. Heterogeneity is essential: it induces variation in marginal propensities to trade, respond to shocks, and value future claims, thereby giving rise to endogenous capital flows and relative price movements. These price adjustments, in turn, feed back into aggregate allocations in a way that reinforces recurrent cycles. In contrast, standard Markovian models with representative or homogeneous agents typically converge to a stationary distribution or steady state, suppressing persistent cyclical behavior.

Our approach highlights that business cycles need not reflect inefficiencies or coordination failures. Instead, they can arise in equilibrium as optimal responses to dispersed incentives and the intrinsic structure of trade in complete markets. The model preserves full Pareto efficiency, and yet generates cycles through endogenous mechanisms alone. This has profound implications for policy: conventional stabilization tools—whether countercyclical or procyclical—may be not only ineffective, but welfare-reducing, as they interfere with efficient allocation paths and create unnecessary fiscal burdens. These findings call for a reassessment of the role of macroeconomic policy in environments where market completeness and agent heterogeneity are first-order features.

6 Appendix

Auxiliary Results

We exhibit a version of the Kunh-Tucker Theorem for the sake of completeness.

Theorem 6.1. *Let* $h : X \subset \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$ *and* $g : X \subset \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{m}}_{+}$ *be concave differentiable column-valued functions, where* X *is an open subset of* $\mathbb{R}^{\mathrm{n}}$*. Suppose that* $\bar{x} \in X$ *satisfies*[24] *the system below for some row vector* $\lambda \in \mathbb{R}^{\mathrm{m}}_{+}$

$$\partial_\iota h(\bar{x}) + \lambda \partial_\iota g(\bar{x}) = 0 \textit{ for } \iota \in \{1, \cdots, \mathrm{n}\} \quad g(\bar{x}) \geq 0 \quad \lambda g(\bar{x}) = 0$$

Then $\bar{x}$ *satisfies*

$$\max\{h(x) \textit{ for all } x \in X \textit{ such that } g(x) \geq 0\}.$$

Proof: Since f and g are concave and the derivative of the concave Lagrangian $\ell : X \subset \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$ defined by

$$\ell(x) = h(x) + \lambda g(x) \text{ for all } x \in \mathbb{R}^{\mathrm{n}}$$

vanishes at $\bar{x}$, it follows that $\bar{x}$ satisfies for $\lambda \in \mathbb{R}^{\mathrm{m}}_{+}$

$$\max\left\{h(x) + \lambda g(x) \text{ for all } x \in X \subset \mathbb{R}^{\mathrm{n}}\right\}.$$

Now, let $x \in X \subset \mathbb{R}^{\mathrm{n}}$ be arbitrary such that $g(x) \geq 0$. We will show that $h(x) \leq h(\bar{x})$. Indeed,

$$h(x) \leq h(x) + \lambda g(x) \leq h(\bar{x}) + \lambda g(\bar{x}) = h(\bar{x}).$$

□

Lemma 6.1. *Consider* $Y_n \subset \mathbb{R}$ *for* $n \leq \mathrm{n}$ *and* $Y = \prod_{n \leq \mathrm{n}} Y_n$*. Let* $l_n : Y_n \times S \to \mathbb{R}$ *be a function concave twice-differentiable on its first coordinate for* $n \leq \mathrm{n}$*. Define* $l : Y \times S \to \mathbb{R}$ *by* $l(y) = \sum_{n \leq \mathrm{n}} l_n(y_n)$*. Suppose that* $\hat{y} : S \to Y$ *satisfies* $D_1 l(\hat{y}(s), s) = 0$ *for all* $s \in S$*. Consider* $(\bar{y}, \bar{\bar{y}}) \in \mathbb{R}^{\mathrm{n}}_{++} \times \mathbb{R}^{\mathrm{n}}_{++}$ *and* $\hat{y}^* : S \to Y$ *defined by*

$$\hat{y}^*(s) = \min\{\max\{\hat{y}(s), \bar{y}\}, \bar{\bar{y}}\} \textit{ for all } s \in S.$$

then for each $s \in S$

$$\hat{y}^*(s) \in \operatorname{argmax}\{l(y, s) \textit{ such that } y \in Y \textit{ and } \bar{y} \leq y \leq \bar{\bar{y}}\}.$$

Proof: Fix $s \in S$. Consider

$$\mathrm{n}^* = \{n \leq \mathrm{n} : \bar{y}_n < \hat{y}_n(s) < \bar{\bar{y}}_n\}$$
$$\bar{\mathrm{n}} = \{n \leq \mathrm{n} : \hat{y}_n(s) \leq \bar{y}_n\} \text{ and } \bar{\bar{\mathrm{n}}} = \{n \leq \mathrm{n} : \hat{y}_n(s) \geq \bar{\bar{y}}_n\}.$$

[24] Write $\partial_\iota h(x)$ for the partial derivative of f with respect to the ι-th variable at point x.

Then $\hat{y}_n^*(s) = \bar{y} \geq \hat{y}_n(s)$ for $n \in \bar{\mathrm{n}}$ and $\hat{y}_n^*(s) = \bar{\bar{y}}_n \leq \hat{y}_n(s)$ for $n \in \bar{\bar{\mathrm{n}}}$. Therefore, the concavity of l_n for each $n \leq \mathrm{n}$ implies[25] that for each $y \in Y$ with $\bar{y} \leq y \leq \bar{\bar{y}}$:

$$\begin{aligned}
l(y,s) - l(\hat{y}^*(s),s) &\leq \sum_{n \leq \mathrm{n}} D_1 l_n(\hat{y}_n^*(s),s)(y_n - \hat{y}_n^*(s)) \\
&= \sum_{n \in \bar{\mathrm{n}}} D_1 l_n(\hat{y}_n^*(s),s)(y_n - \hat{y}_n^*(s)) \\
&+ \sum_{n \in \mathrm{n}^*} D_1 l_n(\hat{y}_n(s),s)(y_n - \hat{y}_n^*(s)) \\
&- \sum_{n \in \bar{\bar{\mathrm{n}}}} D_1 l_n(\hat{y}_n^*(s),s)(\hat{y}_n^*(s) - y_n) \\
&\leq \sum_{n \in \bar{\mathrm{n}}} D_1 l_n(\hat{y}_n(s),s)(y_n - \hat{y}_n^*(s)) \\
&- \sum_{n \in \bar{\bar{\mathrm{n}}}} D_1 l_n(\hat{y}_n(s),s)(\hat{y}_n^*(s) - y_n) \\
&= \sum_{n \leq \mathrm{n}} D_1 l_n(\hat{y}_n(s),s)(y_n - \hat{y}_n^*(s)) \\
&= D_1 l(\hat{y}(s),s)(y - \hat{y}^*(s)) = 0
\end{aligned}$$

and hence $l(y,s) \leq l(\hat{y}^*(s),s)$. Since y was chosen arbitrarily, we get the result.

□

[25] By the Mean Value Theorem.

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