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GENERAL EQUILIBRIUM DYNAMICS FOR INCOMPLETE MARKETS: NUMERICAL EXAMPLES

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GENERAL EQUILIBRIUM DYNAMICS FOR INCOMPLETE MARKETS: NUMERICAL EXAMPLES

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SUMÁRIO

RESUMO

Kubler e Schmedders (2005) mostraram que um equilíbrio competitivo pode estar distante de um equilíbrio exato quando computado usando condições de primeira ordem. Este artigo mostra que um equilíbrio competitivo é implementado por estatísticas recursivas com um espaço de estado mínimo. Este resultado nos permite desenvolver um método alternativo para calcular o equilíbrio sem usar condições de primeira ordem e suposições de suavidade. Calculamos o equilíbrio recursivo por meio de um algoritmo de iteração funcional e mostramos que sua alocação de equilíbrio implementada é arbitrariamente próxima de um equilíbrio competitivo exato. Em particular, simulamos alguns processos de equilíbrio estocástico nos quais os agentes antecipam a incerteza exógena usando algumas regras descritas na literatura de finanças comportamentais. Um importante fato estilizado que é encontrado sugere que mercados incompletos favorecem agentes com expectativas racionais quando o retorno esperado do ativo é alto em relação ao ativo livre de risco.

Palavras-Chave: Sobrevivência; Equilíbrio Recursivo; Aversão à Ambiguidade; Finanças Comportamentais; Noise traders; Mercados Incompletos.

ABSTRACT

Kubler and Schmedders (2005) showed that a competitive equilibrium can be far from an exact equilibrium when computed using first-order conditions. This paper shows that a competitive equilibrium is implemented by recursive statistics with a minimal state space. This result allows us to develop an alternative method for computing the equilibrium without using first order conditions and smoothness assumptions. We compute the recursive equilibrium through a functional iteration algorithm and show that its implemented equilibrium allocation is arbitrarily close to an exact competitive equilibrium. In particular, we simulate some stochastic equilibrium processes in which agents anticipate exogenous uncertainty using some rules described in the literature of behavioral finance. An important stylized fact that is found suggests that incomplete markets favor agents with rational expectations when the expected asset return is high in relation to the risk-free asset.

Keywords Survival; Recursive Equilibrium; Ambiguity Aversion; Behavioral finance; Noise Traders; Incomplete Markets.

JEL Code: D58, D52

1 Introduction

Recursive methods constitute a powerful approach towards dynamic models due to their described focus on a trade off between the current period's utility and a continuation value for utility in all future periods. Therefore, the computation of the recursive equilibrium with heterogeneous agents is an important tool of analysis in the economics literature. In general, for differentiable policy functions, the equilibrium is computed using the first order conditions as in Schmedders (1998) or Judd et al. (2000), among others. Even in the case of non-differentiable policy functions, we can find alternative numerical methods such as in Brumm and Grill (2014) and Fella (2014).

Unfortunately, Kubler and Schmedders (2005) show that the competitive equilibrium computed through the first order conditions might be very far from the exact equilibrium. Kubler (2011) provides conditions for avoiding this problem although his solution seems quite restrictive. In general, the models with heterogeneous agents can generate corner conditions, precluding the use of first order conditions to compute the sequential equilibrium. Brumm and Grill (2014) and Fella (2014) provide interesting methods when corner conditions are finite and under some regularity conditions, despite using the first order conditions in the interior points. Thus, these results are subject to the same criticism made by Kubler and Schmedders (2005) and, therefore, the sequential equilibrium can be computed far from an exact one. In this paper, we use a parsimonious method for computing models with heterogeneous agents without using the first order conditions. As an application, we provide numerical examples elucidating how markets with heterogeneous beliefs select for particular types of investors in an environment with incomplete markets, common discount factor, and long lived assets without short sales.

There is a long history in economics of using market selection arguments in defense of rationality hypotheses. Traditional economic theory argues that markets favor rational over irrational investors and as a consequence only the rational ones survive (Alchian, 1950; Friedman, 1953).[1] For instance, inaccurate beliefs, as a possible form of irrationality, induce agents to make wrong investment decisions and thus, lose their wealth in favor of agents with more accurate beliefs. Therefore, the later accumulate all the wealth and drive the ones with inaccurate beliefs out of the market.

De Long et al. (1990) analyze selection over traders who are subjective expected utility maximizers with differing beliefs. In their paper, prices are set exogenously and traders whose beliefs reflect irrational overconfidence can eventually dominate the asset market. This result appears to contradict the conclusions of Alchian and Friedman. But, as prices are given as exogenous, these traders are not really trading with each other; if they were, then whenever traders with incorrect beliefs dominate the market, prices would reflect their beliefs and rational traders might be able to take advantage of them. The intuitions of Alchian and Friedman were correct in an economy with complete markets, endogenous prices and traders with a common discount factor. Sandroni (2000) shows that traders with rational expectations dominate the market in a Lucas tree model. Blume and Easley (2006) indicate that this result holds in any Pareto optimal allocation and thus, in any equilibrium that originates from a dynamically complete market. Unfortunately, there are no results regarding survival in models with incomplete markets and heterogeneous beliefs. Blume and Easley (2006) exhibit some counterexamples for incomplete markets

[1] See also Sandroni (2000), Blume and Easley (2006) for instance.

in which rational traders are driven out of the market. This justifies the need to compute the competitive equilibrium in such markets in order to understand the dynamics of the equilibrium stochastic process.

In this paper we provide numerical examples elucidating how markets select for particular types of investors with heterogeneous beliefs. Basically, we give examples in the models with noise traders, ambiguity aversion, prospect theory and behavioral finance. Roughly speaking, the noise traders are those with accurate beliefs in the mean of dividends and inaccurate beliefs in its variance. Agents with ambiguity aversion have a maxmin expected utility over all probability distributions they suppose possible under their beliefs. Prospect theory is an approach in which agents overweight outcomes that are considered certain relative to uncertain outcomes. The example in behavioral finance assumes that agents estimate the exogenous shock with beliefs characterized by a Markov switching regime, which is a kind of belief consistent with price underreaction to positive shocks and price overreaction to negative shocks. All numerical results confirm Beker and Chattopadhyay (2010), which state that in environments with two agents, one asset and heterogeneous beliefs, at least one must have null asymptotic consumption, provided that agents do not have periodic income.

We show that a recursive equilibrium with the state space consisting of the previous portfolio distribution and the current state of nature implements a sequential equilibrium. This result allows us to numerically obtain a competitive equilibrium by computing the recursive equilibrium. We make it by solving functional equations through an iteration algorithm in a manner similar to that found in Judd (1998). There are two functional equations in this problem: the first finds the value and policy functions determined by agents' optimization problem through the Bellman equation; the second is given by the market clearing conditions satisfied by the price transition functions. The main difference in this method compared to Judd et al. (2000) consists in the way to compute the sequential equilibrium. Instead of directly using the Euler equations, we initially fix price and portfolio transition functions suitable to the convergence, thus finding a new portfolio transition through the policy functions. Using the approximate market clearing conditions and the law of demand,[2] we get the new transition price, continuing the process until the convergence for the recursive equilibrium. The sequential equilibrium is obtained by a suitable iteration of the later.

The recursive approach is based on the existence results given in Duffie et al. (1994), Kubler and Polemarchakis (2004) and Kubler and Schmedders (2002). Duffie et al. (1994) show the existence of a recursive equilibrium with a large state space and Kubler and Polemarchakis (2004) show the existence of an approximate recursive equilibrium with a minimal state space. Kubler and Schmedders (2002) argue that uniqueness of the sequential equilibrium assures the existence of a recursive equilibrium with minimal state space. We obtain convergence in the numerical examples assuming that the state space is the previous period portfolio distribution and the current state of nature. This state space is actually minimal since a reallocation of the initial portfolio distribution in a setting with heterogeneous risk aversion would typically change the previous equilibrium price.

The paper is structured as follows. In Section 2 we establish the model. Section 3 defines the equilibrium concepts and exhibits some results. In Section 4, we present the numerical results. Conclusions are given in Section 5. Section 6 is the appendix.

[2] We assume that the good is normal, and hence its demand is decreasing on its price.

2 The model

In this section we establish the primitives of the model, including environment characteristics and agents' features.

2.1 Definitions

Suppose that there exist finite types in the economy denoted by the set $\mathcal{I} = \{1, ..., I\}$ and such that each type $i \in \mathcal{I}$ has a continuum of agents trading in a competitive environment. Time is indexed by t in the set $\mathbb{N} = \{1, 2, ...\}$. In this model the uncertainty is exogenous, in the sense of being independent of agents' actions. Each agent knows the whole set of possible states of nature and trades contingent claims. Let Z be a topological space containing all states of nature and $\mathscr{Z}$ its Borel subsets. Denote by $(Z_t, \mathscr{Z}_t)$ a copy of $(Z, \mathscr{Z})$ for all $t \in \mathbb{N}$. Exogenous uncertainty is described by the streams $z^t = (z_1, ..., z_t) \in Z_1 \times \cdots \times Z_t := Z^t$ for all $t \in \mathbb{N}$. Write $z_2^t = (z_2, ..., z_t)$ and $Z_2^t = Z_2 \times \cdots \times Z_t$. The set Z contains the exogenous variables on which the beliefs will be defined.

There are one good and a finite set $\mathcal{H} = \{1, ..., H\}$ of long lived real assets in net supply equal to one and with dividends measurable bounded functions $\hat{d} : Z \to \mathbb{R}^H_{++}$ given in units of the good. One unit of asset h in a state z gives the right to receive $d_h(z)$ units of the good in the next period. Denote by $\Theta^i \subset \mathbb{R}^H_+$ for all $i \in \mathcal{I}$ the convex set where asset choices are defined and $C^i \subset \mathbb{R}_+$ the convex set where agent i's consumption is chosen. Observe that we are not allowing for short-sales. Define the symbol without upper index as the Cartesian product. For example, write $C = \prod_{i \in \mathcal{I}} C^i$.

Denote by $Q = \{(q^c, q^a) \in \mathbb{R}_+ \times \mathbb{R}^H_+ : q^c + \sum_{h \in \mathcal{H}} q^a_h = 1\}$ the set where prices are defined and write $Q^\circ = Q \cap \mathbb{R}^{H+1}_{++}$. The symbol $q = (q^c, q^a) \in Q$ stands for consumption and asset prices respectively. Write $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^H$ and $\mathbf{1} \cdot d(z) = \sum_{h \in \mathcal{H}} d_h(z)$.

Write $\overline{\Theta} = \{\bar{\theta} \in \Theta : \sum_{i \in \mathcal{I}} \bar{\theta}^i_h = 1 \text{ for all } h \in \mathcal{H}\}$. An element $\bar{\theta} \in \overline{\Theta}$ stands for the asset distribution of the agents. Since each type $i \in \mathcal{I}$ has a continuum of agents, we assume that choices are identical among agents of same type.[3]

Let $S = \overline{\Theta} \times Z$ be the space of state variables endowed with the product topology and the Borel sigma-algebra $\mathscr{S}$. Write the set of all measurable functions $\hat{q} : S \to Q$ by $\widehat{Q}$ and the set of all measurable functions $\hat{q} : S \to Q^\circ$ by $\widehat{Q}^\circ$.

Every Cartesian product of topological spaces is endowed with the product topology. In particular, $\widehat{Q}$ is endowed with the sup norm metric.

The instantaneous utility is a bounded continuous real valued function $u^i : \mathbb{R}_+ \to \mathbb{R}_+$ strictly concave and strictly increasing for all $i \in \mathcal{I}$.

2.2 Agents' features

Agents' beliefs at initial period are characterized by the continuous[4] transitions $\{\mu^i_t : Z \to \text{Prob}(Z^t_2)\}_{t \geq 2}$, forecasting each future period t exogenous streams of states of nature given the realization of the initial period state of nature z_1. We suppose that these beliefs are *predictive* in the context of Blackwell and

[3]Therefore we use a symmetric equilibrium approach in the model.

[4]The set $\text{Prob}(Z^t_2)$ is endowed with the weak topology.

Dubins (1962) with continuous probability transition rules[5] $\lambda^i : Z \to \text{Prob}(Z)$. That is, the measure μ_t^i satisfies[6] for a rectangle $A_2 \times ... \times A_t$:

$$\mu_t^i(z_1)(A_2^t) = \int_{A_2} \cdots \int_{A_t} \lambda(z_{t-1}, dz_t) \cdots \lambda(z_1, dz_2). \tag{1}$$

Moreover we assume to simplify notation that agents use only one period backward to estimate future states of the economy. More formally, agents suppose that the exogenous uncertainty is governed by a Markov stochastic process $\{\hat{z}_t : \Omega \to Z\}_{t\in\mathbb{N}}$ defined on a subjective probability space $(\Omega, \Sigma, \mathbb{P}^i)$. They choose probability transition rules $\{\lambda^i\}_{i\in\mathcal{I}}$ satisfying for each $A \in \mathscr{Z}$ the relation $\lambda^i(\hat{z}_t(\omega), A) = \mathbb{P}^i(\hat{z}_{t+1} \in A|\mathscr{F}_t)(\omega)$ for all ω in a set $\Omega' \in \Sigma$ such that $\mathbb{P}^i(\Omega') = 1$ where[7] $\mathscr{F}_t = \sigma(\hat{z}_t)$, that is, $\lambda^i(\hat{z}_t(\cdot), \cdot)$ is a conditional distribution[8] of $\hat{z}_{t+1}$ given $\hat{z}_t$ for all $t \in \mathbb{N}$. Using Lemma 6.1 in appendix we conclude that

$$\mu_t^i(\hat{z}_1(\omega), A_2^t) = \mathbb{P}^i(\hat{z}_2^t \in A_2^t|\mathscr{F}_1)(\omega).$$

Assume that agents make plans at the initial period contingent to all possible future trajectories of exogenous variable, since they do not perfectly anticipate them in the future. We also suppose that the realization of the random variable $\hat{z}_1$ is observed at the initial period.

Definition 2.1. *An agent i's consumption and portfolio plans are defined as a stream of measurable functions* $\boldsymbol{c}_t^i : Z^t \to C^i$ *and* $\boldsymbol{\theta}_t^i : Z^t \to \Theta^i$ *respectively, for all* $t \in \mathbb{N}$. *We denote a consumption plan as* $\boldsymbol{c}^i := \{\boldsymbol{c}_t^i\}_{t\in\mathbb{N}}$ *and a portfolio plan as* $\boldsymbol{\theta}^i := \{\boldsymbol{\theta}_t^i\}_{t\in\mathbb{N}}$. *Moreover, write* $\boldsymbol{C}^i$ *the set of all consumption plans and* $\boldsymbol{\Theta}^i$ *the set of all portfolio plans.*

At each period t, the amount $\boldsymbol{c}_t^i(z^t)$ can be interpreted as the value planned initially for period t consumption if $z^t = (z_1, ..., z_t)$ will be the partial history of the state variable observed at periods $1, ..., t$. The asset plan $\boldsymbol{\theta}_t^i$ has analogous interpretation. The intertemporal consistency assures that agents will choose their plans in the future.

Definition 2.2. *Let* $B^i : \Theta^i \times Z \times Q \to C^i \times \Theta^i$ *be defined as*

$$B^i(\theta_-^i, z_1, q) = \{(c^i, \theta^i) \in C^i \times \Theta^i : q^c c^i + q^a \theta^i \leq (q^a + q^c d(z_1))\theta_-^i\}.$$

Let $\boldsymbol{Q}$ *be the set of all stream of contingent prices* $\boldsymbol{q} = \{\boldsymbol{q}_t : Z^t \to Q\}_{t\in\mathbb{N}}$. *A plan* $(\boldsymbol{c}^i, \boldsymbol{\theta}^i)$ *is feasible from* $(\theta_-^i, z_1, \boldsymbol{q})$ *for agent i if*

$$(\boldsymbol{c}_t^i(z^t), \boldsymbol{\theta}_t^i(z^t)) \in B^i(\boldsymbol{\theta}_{t-1}^i(z^{t-1}), z_t, \boldsymbol{q}_t(z^t)) \text{ for all } z^t \in Z^t$$

and for all $t \in \mathbb{N}$ *where we define by convention that* $\boldsymbol{\theta}_0^i = \theta_-^i$.

Denote by $\boldsymbol{F}^i(\theta_-^i, z_1, \boldsymbol{q})$ *the set of all feasible plans from* $(\theta_-^i, z_1, \boldsymbol{q})$.

[5] With a slight abuse of notation, write $\lambda^i(z_1)(A)$ as $\lambda^i(z_1, A)$ for each $(z_1, A) \in Z \times \mathscr{Z}$.

[6] See Stokey et al. (1989) Chapters 8 and 9 for details about the construction of a probability measure based on the composition of probability transition rules and results about expectations over this measure.

[7] That is, $\mathscr{F}_t$ is the smallest σ-algebra in which $\hat{z}_t$ is measurable.

[8] Following the definition given in Billingsley (1968).

Now we can define the expected utility.

Definition 2.3. *Define the agent i's expected utility $\boldsymbol{U}^i : \boldsymbol{C}^i \times Z \to \mathbb{R}$ for consuming $\boldsymbol{c}^i$ given the state $z_1 \in Z$ as:*

$$\boldsymbol{U}^i(\boldsymbol{c}^i, z_1) = u^i(\boldsymbol{c}_1^i(z_1)) + \sum_{t\geq 2} \int_{Z_2^t} \beta^{t-1} u^i(\boldsymbol{c}_t^i(z_1, z_2^t))\mu_t^i(z_1, dz_2^t).$$

Definition 2.4. *Define the value function $\hat{v}^i : \Theta^i \times Z \times \boldsymbol{Q} \to \mathbb{R}$ by:*

$$\hat{v}^i(\theta_-^i, z_1, \boldsymbol{q}) = \sup\{\boldsymbol{U}^i(\boldsymbol{c}^i, z_1) : (\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{F}^i(\theta_-^i, z_1, \boldsymbol{q})\} \tag{2}$$

and the optimal demand $\delta^i : \Theta^i \times Z \times \boldsymbol{Q} \to \boldsymbol{C}^i \times \boldsymbol{\Theta}^i$ by:

$$\delta^i(\theta_-^i, z_1, \boldsymbol{q}) = \text{argmax}\{\boldsymbol{U}^i(\boldsymbol{c}^i, z_1) : (\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{F}^i(\theta_-^i, z_1, \boldsymbol{q})\}$$

3 Recursive and sequential equilibrium

In this section we define these two concepts of equilibrium and show the relation between them. The recursive equilibrium is defined as in Judd et al. (2000) but without using the first order conditions of optimality.

The next definition specifies the sequential equilibrium of the economy.

Definition 3.1. *For $t \in \mathbb{N}$, consider the following measurable*

1. $\boldsymbol{q}_t : Z^t \to Q^\circ$ *contingent prices;*
2. $\mathbf{c}_t : Z^t \to C$ *contingent consumption allocation;*
3. $\boldsymbol{\theta}_t : Z^t \to \Theta$ *contingent portfolio allocation*

and $\bar{\theta} \in \overline{\Theta}$ the initial portfolio distribution. Then these allocations and prices constitute an equilibrium for $\mathcal{E}$ if satisfy for all $z^t \in Z^t$ and all $t \in \mathbb{N}$:

1. *optimality:* $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \delta^i(\bar{\theta}^i, z_1, \boldsymbol{q})$*;*
2. *asset markets clear:* $\sum_{i\in\mathcal{I}} \boldsymbol{\theta}^i(z^t) = \mathbf{1}$*;*
3. *good markets clear:* $\sum_{i\in\mathcal{I}} \boldsymbol{c}^i(z^t) = \mathbf{1} \cdot d(z_t)$*.*

We introduce now the concept of recursive equilibrium and show in the appendix that it implements the sequential equilibrium of the economy. Observe that the value function depends on the portfolio transition of the economy and the price transition function. This assures that agents have common expectations[9] in the sequential equilibrium implemented by the recursive equilibrium. Indeed, as we will show below, agents accurately anticipate the endogenous variables of the sequential equilibrium implemented by the recursive equilibrium. Write $\widehat{\Theta}$ the space of all continuous functions $\hat{\theta} : S \to \overline{\Theta}$. A well

[9] We say that agents have common expectations when each agent associate the same price to the same realization of the exogenous event. See Radner (1972) for details about this concept.

known result states that there exists a bounded continuous value function $v^i : \Theta^i \times S \times \widehat{Q}^\circ \times \widehat{\Theta} \to \mathbb{R}$ for $i \in \mathcal{I}$ satisfying the Bellman equation:

$$v^i(\theta^i_-, s, \hat{q}, \hat{\theta}) = \sup \left\{ u(c^i) + \beta \int_Z v^i(\theta^i, (\hat{\theta}(s), z'), \hat{q}, \hat{\theta}) \lambda^i(z, dz') \right\} \tag{3}$$

over all $(c^i, \theta^i) \in B^i(\theta^i_-, z, \hat{q}(s))$. Indeed, write $\mathbb{V}$ the space of all bounded value functions $v^i : \Theta^i \times S \times \widehat{Q}^\circ \times \widehat{\Theta} \to \mathbb{R}$ endowed with the sup norm and consider the operator $T^i : \mathbb{V} \to \mathbb{V}$, defined by

$$T^i(v^i)(\theta^i_-, s, \hat{q}, \hat{\theta}) = \sup \left\{ u(c^i) + \beta \int_Z v^i(\theta^i, (\hat{\theta}(s), z'), \hat{q}, \hat{\theta}) \lambda^i(z, dz') \right\} \tag{4}$$

over all $(c^i, \theta^i) \in B^i(\theta^i_-, z, \hat{q}(s))$. Clearly, T satisfies Blackwell's sufficient conditions for a contraction and hence has a fixed point. See Stokey et al. (1989) and the Berge Maximum Theorem (Aliprantis and Border, 1999) for further details.

We stand out the argmax of the type i Bellman equation (3) on the following definition.

Definition 3.2. *Define the agent i's consumption and portfolio policy correspondence $\tilde{x}^i : \Theta^i \times S \times \widehat{Q}^\circ \times \widehat{\Theta} \to C^i \times \Theta^i$ with $\tilde{x}^i = (\tilde{c}^i, \tilde{\theta}^i)$ as*

$$\tilde{x}^i(\theta^i_-, s, \hat{q}, \hat{\theta}) = \operatorname{argmax} \left\{ u(c^i) + \beta \int_Z v^i(\theta^i, (\hat{\theta}(s), z'), \hat{q}, \hat{\theta}) \lambda^i(z, dz') \right\}$$

over all $(c^i, \theta^i) \in B^i(\theta^i_-, z, \hat{q}(s))$;

Notice that the policy correspondence satisfy

$$\tilde{x}^i(\theta^i_-, s, \hat{q}, \hat{\theta}) \subset B^i(\theta^i_-, z, \hat{q}(s)) \text{ for all } (\theta^i_-, s, \hat{q}, \hat{\theta}) \in \Theta^i \times S \times \widehat{Q}^\circ \times \widehat{\Theta}. \tag{5}$$

Definition 3.3. *We say that the economy has a recursive equilibrium if there exist functions $\hat{c}^i : S \to C^i$, $\hat{\theta}^i : S \to \Theta^i$ for $i \in \mathcal{I}$ and $\hat{q} : S \to Q^\circ$ satisfying for each $s = (\bar{\theta}, z) \in S$*

1. *optimality: $(\hat{c}^i(s), \hat{\theta}^i(s)) \in \tilde{x}^i(\bar{\theta}^i, s, \hat{q}, \hat{\theta})$ for all $i \in \mathcal{I}$;*
2. *consumption market clearing: $\sum_{i \in \mathcal{I}} \hat{c}^i(s) = \mathbf{1} \cdot d(z)$;*
3. *asset market clearing: $\sum_{i \in \mathcal{I}} \hat{\theta}^i(s) = \mathbf{1} \in \mathbb{R}^H$.*

Definition 3.4. *We say that the functions $\hat{c} : S \to C$, $\hat{\theta} : S \to \Theta$ and $\hat{q} : S \to Q$ implement the measurable stream $\{\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t \in \mathbb{N}}$ starting from $\bar{\theta} \in \overline{\Theta}$ if for all $z^t \in Z^t$ and all $t \in \mathbb{N}$*

$$\boldsymbol{q}_1(z_1) = \hat{q}(\bar{\theta}, z_1), \quad \boldsymbol{\theta}^i_1(z_1) = \hat{\theta}^i(\bar{\theta}, z_1), \quad \boldsymbol{c}^i(z_1) = \hat{\theta}^i(\bar{\theta}, z_1)$$

and recursively for $t \geq 2$

$$\boldsymbol{c}^i_t(z^t) = \hat{c}^i(\boldsymbol{\theta}_{t-1}(z^{t-1}), z_t) \qquad \boldsymbol{\theta}^i_t(z^t) = \hat{\theta}^i(\boldsymbol{\theta}_{t-1}(z^{t-1}), z_t) \tag{6}$$

for $i \in \mathcal{I}$ *and*

$$\boldsymbol{q}_t(z^t) = \hat{q}(\boldsymbol{\theta}_{t-1}(z^{t-1}), z_t). \tag{7}$$

The next result assures that the recursive equilibrium can actually be used to construct the sequential equilibrium.

Theorem 3.5. *If* $(\hat{c}, \hat{\theta}, \hat{q})$ *is a recursive equilibrium then its implemented process* $\{\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t\in\mathbb{N}}$ *starting from* $\bar{\theta} \in \overline{\Theta}$ *is a sequential equilibrium of the economy with initial asset holdings* $\bar{\theta} \in \overline{\Theta}$.

Proof: See theorem 6.2 in the appendix. □

The existence result below is found in Kubler and Schmedders (2002). Magill and Quinzii (1994) have shown the existence of the sequential equilibrium.

Theorem 3.6. *Suppose that there exists a unique sequential equilibrium. Then there exists a recursive equilibrium with the minimal state space* S.

4 Numerical simulations

The numerical method used here is similar to that found in Judd (1998) and Judd et al. (2000). Basically, we proceed iterating functions recursively to find the solution of a certain functional equation as a fixed function on the limit.[10]

To clarify ideas, given a price $\hat{q}$ and a transition $\hat{\theta}$, write

$$\hat{\theta}'(\bar{\theta}^i, \bar{\theta}, z, \hat{q}, \hat{\theta}) = (\tilde{\theta}^i(\bar{\theta}^i, \bar{\theta}, z, \hat{q}, \hat{\theta}))_{i\in\mathcal{I}} \text{ for all } (\bar{\theta}, z) \in S$$

where $\tilde{\theta}^i : \Theta^i \times \overline{\Theta} \times Z \times \widehat{Q}^\circ \times \widehat{\Theta} \to \Theta^i$ is the argmax of the type i Bellman equation (3). We compute first the value function $v^i(\,\cdot\,, \,\cdot\,, \hat{q}, \hat{\theta})$ on a finite set, say, $gridS \subset S$ through the standard Bellman method iterating value functions given an initial function v_0^i. After that, we compute the asset excess demand

$$\hat{x}^a(\bar{\theta}, z) = \left(\sum_{i\in\mathcal{I}} \tilde{c}^i(\bar{\theta}^i, \bar{\theta}, z, \hat{q}, \hat{\theta}) - \mathbf{1}\cdot d(z), \sum_{i\in\mathcal{I}} \tilde{\theta}^i(\bar{\theta}^i, \bar{\theta}, z, \hat{q}, \hat{\theta}) - \mathbf{1} \right) \text{ for all } (\bar{\theta}, z) \in S,$$

which is the argmax of the value function $v^i(\,\cdot\,, \,\cdot\,, \hat{q}, \hat{\theta})$ on $gridS$. Choosing a price $\hat{q}' = \hat{q} + \Delta\hat{q}' \in \widehat{Q}^\circ$ where $\Delta\hat{q}$ is an increment proportional to excess demand function $\hat{x}^a$, we compute again $\hat{\theta}'' = (\tilde{\theta}^i(\bar{\theta}^i, \bar{\theta}, z, \hat{\theta}', \hat{q}'))_{i\in\mathcal{I}}$ for all $(\bar{\theta}, z) \in gridS$ and repeat the previous step until the desired precision. The proportionality constant of the increment $\Delta\hat{q}$ is chosen appropriately to ensure the speed of convergence. Notice that the convergence of the algorithm implies that the increment $\Delta\hat{q}'$ is near zero and therefore the excess demand $\hat{x}^a$ also tends to zero in the limit.

In the following sections we suppose that $H = 1$ and that the dividend function $d : Z \to \mathbb{R}_{++}$ is increasing on $Z = \{1, 2\}$ or $Z = \{1, 2, 3\}$. The utility function is given by $u^i(c) = 2c^{1/2}$ for all $i \in \mathcal{I}$. The following definition is similar to that found in Blume and Easley (1982) and characterizes the survival and market dominance in the economy.

[10] On the sup norm.

Definition 4.1. *Consider $\{\hat{z}_t\}_{t\in\mathbb{N}}$ a stochastic process governing exogenous uncertainty and defined on a underlying objective probability space $(\Omega, \Sigma, \mathbb{P})$.*

1. *We say that an agent i survives if there exists $A \in \Sigma$ with $\mathbb{P}(A) > 0$ and such that $\liminf_t \theta_t^i(\hat{z}^t(\omega)) > 0$ for all $\omega \in A$.*

2. *We say that an agent i dominates the market if there exists $A \in \Sigma$ with $\mathbb{P}(A) = 1$ and such that $\liminf_t \theta_t^i(\hat{z}^t(\omega)) = 1$ for all $\omega \in A$.*

3. *We say that an agent i is driven out of the market if there exists $A \in \Sigma$ with $\mathbb{P}(A) = 1$ and such that $\limsup_t \theta_t^i(z^t) = 0$ for all $\omega \in A$.*

In the next sections, we infer the survival through the mean estimator for $i \in \mathcal{I}$

$$\sum_{r\leq 5000} \boldsymbol{\theta}_t^i(\hat{z}_r^t(\omega))/5000 \text{ for } t = 1, 2, ..., T$$

for $r = 5000$ trajectories of the stochastic process $\hat{z}_r^t(\omega) = (\hat{z}_{r1}(\omega), \hat{z}_{r2}(\omega), ..., \hat{z}_{rt}(\omega))$ which in general converges $\mathbb{P}$-almost everywhere as T is large. We can also infer the convergence by observing for each fixed $z \in Z$ the policy function graph $\bar{\theta} \to \hat{\theta}(\bar{\theta}, z)$ which describes the dynamics of each trajectory of $\{\boldsymbol{\theta}_t^i(\hat{z}_r^t(\,\cdot\,))\}_{t\leq T}$.

4.1 Ambiguity Aversion

This section refers to a class of preferences first envisioned by Knight (2012) and more recently formalized by Gilboa (1987) and Gilboa and Schmeidler (1989). Investors who are ambiguity averse act as if they believe that the realization of uncertainty could possibly be described by one of a number of different probability distributions. Therefore, they make investment decisions which maximize the minimum expected utility across such distributions, that is, preferences can be characterized by a *maxmin expected utility*. This approach allows for the precise description of the difference between risk and uncertainty. In a setting involving risk, the objective probability is available to guide choice and in an environment with ambiguity aversion, the information is too imprecise to be summarized adequately by one probability.

There are now available several extensions of preferences over lotteries based on an expected utility representation as in Savage (1954) that admit a distinction between risk and uncertainty. One, due to Bewley (2002), drops Savage's assumption that preferences are complete and adds a model of the "status quo". An alternative direction, due to Gilboa and Schmeidler (1989), is to weaken Savage's Sure-Thing Principle. The consequence for the representation of preferences and beliefs is that Savage's single prior is replaced by a set of priors.

In case of complete markets, Condie (2008) investigates the long run wealth of investors in an asset market populated by agents with heterogeneous preferences over risk and ambiguity. If ambiguity averse investors always believe that the true distribution could be wrong in many possible directions, then a necessary condition for their survival is that the market exhibits no aggregate risk, a condition not met by many asset pricing models of interest. However Condie (2008) shows that there do exist markets in which ambiguity averse investors survive. The second example in this section suggests that this result can be extended to the case of incomplete markets.

Current computational methods for models with heterogeneous agents (Brumm and Grill, 2014; Fella, 2014) are not suitable for models with ambiguity aversion because the functional representation of agent's preferences is not differentiable on a large subset of the domain of recursive functions. This characterizes a motivation for the following simulation.

Consider an environment with incomplete markets and two types $i \in \{j, k\}$. Suppose that agent j has ambiguity aversion and agent k has rational expectations. Let $P^j \subset \text{Prob}(Z)$ be the set of priors[11] independent of the current state of nature in which agents construct their preferences. The Bellman equation characterizing agent j's optimal problem is given by

$$v^j(\theta^j_-, s, \hat{q}, \hat{\theta}) = \max \left\{ u(c^j) + \beta \min \left\{ \int_Z v^j(\theta^j, (\hat{\theta}(s), z'), \hat{q}, \hat{\theta}) p(dz') : p \in P^j \right\} \right\}$$

over all $(c^j, \theta^j) \in B^j(\theta^j_-, z, \hat{q}(s))$.

Remark 4.1. Epstein and Wang (1994) and Epstein and Schneider (2003) show that the recursive equilibrium implements the sequential equilibrium under the rectangularity condition.[12]

4.1.1 Linear kernel P^j

Suppose the set of priors $p = (p_1, p_2)$ defined over the set $Z = \{z_1, z_2\}$ and given by an interval on the probability of the state z_1, that is, $p_1 \in [p_1^l, p_1^h]$ where p_1^l is the lower level of probability of z_1 and p_1^h is the higher level of probability of z_1 that is:

$$P^j = \{p \in \mathbb{R}^2_+ : p_1 + p_2 = 1 \text{ and } p_1 \in [p_1^l, p_1^h]\}.$$

We fix the lower probability belief on the state one: $p_1^l = 0.2$ and the true probability $\bar{p} = (0.4, 0.6)$. Moreover, we simulate the equilibrium for $p_1^h = 0.25;\ 0.3;\ 0.4;\ 0.5;\ 0.6$ and 0.8. Therefore, we have $0.2 = p_1^l \leq p_1 \leq p_1^h$ for all $p \in P_j$ and $\lambda^k(z) = \bar{p} = (0.4, 0.6)$ for $z \in Z$, that is, $\{\hat{z}_t\}_{t \in \mathbb{N}}$ is i.i.d with distribution $\bar{p}$. Figure 1 elucidates the result below.

Numerical Result 4.2. *Let $\bar{p}$ be the true probability and write $p_1^* = \max\{p_1 : p \in P^j\}$. If $p_1^* < \bar{p}_1$ then agents with ambiguity aversion dominate the market on all simulations. Otherwise they are driven out of the market.*

4.1.2 ϵ-contamination

This example was defined first by Huber (1965) and is used in recent papers in the literature of ambiguity aversion as in Epstein and Wang (1994). Basically, the definitions above are the same and the only difference here is the family of priors $P^j(\epsilon, \bar{p})$ which is defined as :

$$P^j(\epsilon, \bar{p}) = \{p \in \text{Prob}(Z) : p = (1 - \epsilon)\bar{p} + \epsilon p' \text{ for some } p' \in \text{Prob}(Z)\}.$$

[11] Also called probability kernel.

[12] See Condie (2008) for the concept of rectangularity.

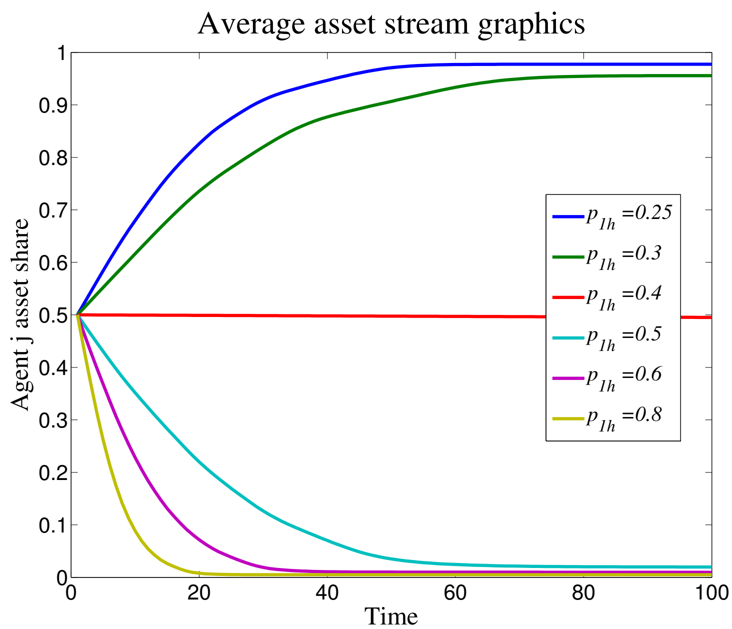

Figure 1: Graph of $\sum_{r\leq 5000}\boldsymbol{\theta}_t^j(\hat{z}_r^t(\omega))/5000$ for $r=5000$ trajectories of $(z_1,z_2,\ldots,z_T)$, $T=100$ and $\lambda^k(z)=p=(0.4,0.6)$ for each $z\in\{z_1,z_2\}$.

and called ϵ-contamination set with respect to the true probability $\bar{p}$. We simulate the portfolio dynamics for $\epsilon\in\{1/5,1/2,4/5\}$ and show this average over $t=5000$ trajectories in the Figure 2.

Notice that agents with rational expectations dominate the market in all simulations. Furthermore, when ϵ increases the ambiguity aversion also increases. Therefore, the convergence of agent j's average portfolio for $\epsilon=4/5$ is faster than the convergence for $\epsilon=1/5$. Observe that in this case the true distribution could be wrong in all possible directions under agent i's beliefs. Figure 3 clarifies the dynamics for $\epsilon\in\{1/5,1/2,4/5\}$ and we can enunciate the following result

Numerical Result 4.3. *Agents with ϵ-contamination preferences with respect to the true probability are driven out of the market by the agents with Rational Expectations for all simulations.*

4.2 Behavioral Finance

This example is based on Barberis et al. (1998). In their paper, they present a parsimonious model of investor sentiment, or of how investors form beliefs, which is consistent with the empirical findings. Basically, they model the following regularities: underreaction of stock prices to news like earnings announcements[13] and overreaction of stock prices to a series of bad news.[14] As a consequence, news is only slowly incorporated into prices, which tend to exhibit positive autocorrelations over these horizons. The overreaction evidence shows that over longer horizons of perhaps 3-5 years, asset prices overreact to consistent patterns of news pointing in the same direction. Assets that have had a long record of good

[13] That is, stock prices do not immediately adjust to the increase of the level of dividends.

[14] That is, stock prices do not immediately adjust to the decrease of the level of dividends. They use the U.S. data to find empirically these stylized facts.

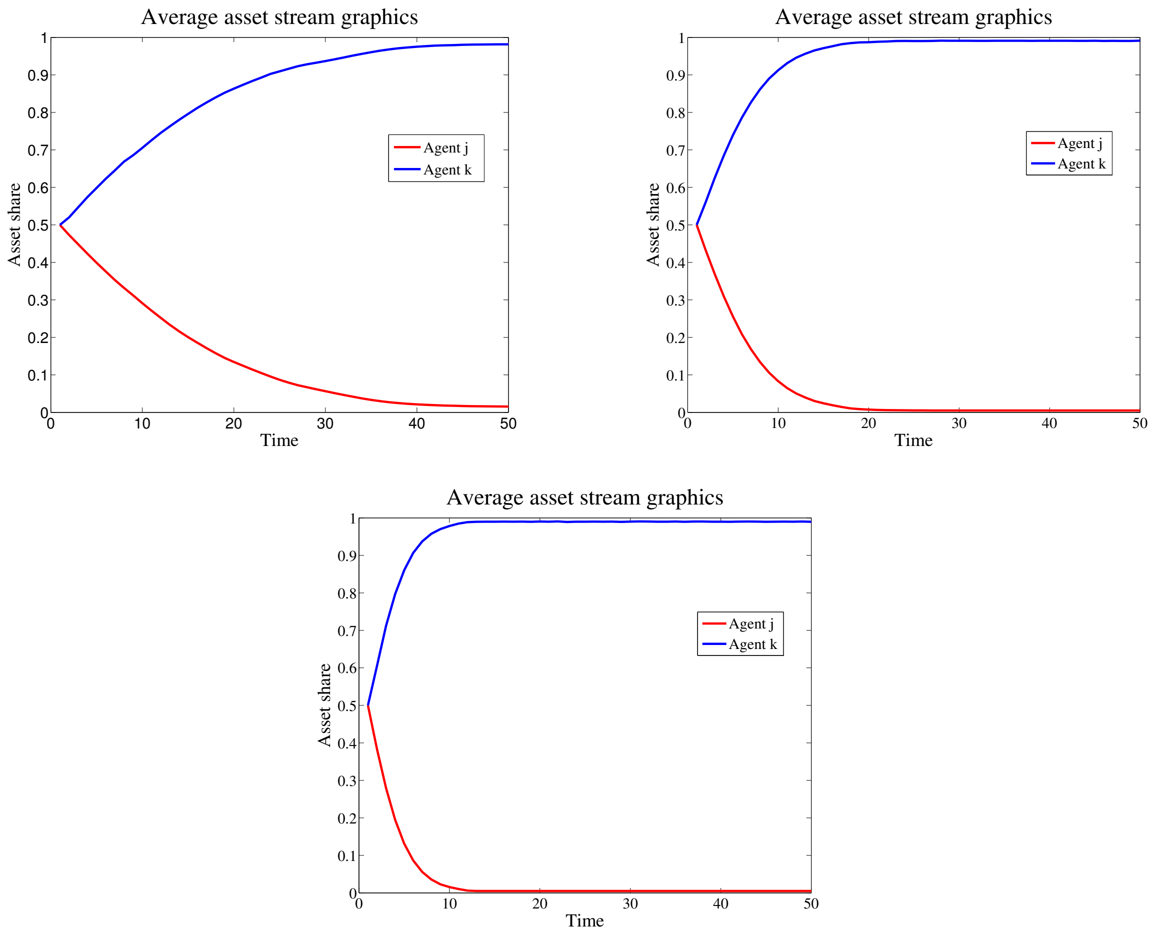

Figure 2: Graph of $\sum_{r\leq 5000} \boldsymbol{\theta}_t^j(\hat{z}_r^t(\omega))/5000$ for $t = 5000$ trajectories of $(z_1, z_2, ..., z_T)$, $T = 50$ and $\epsilon = 1/5$ on the left $\epsilon = 1/2$ on the right and $\epsilon = 4/5$ below.

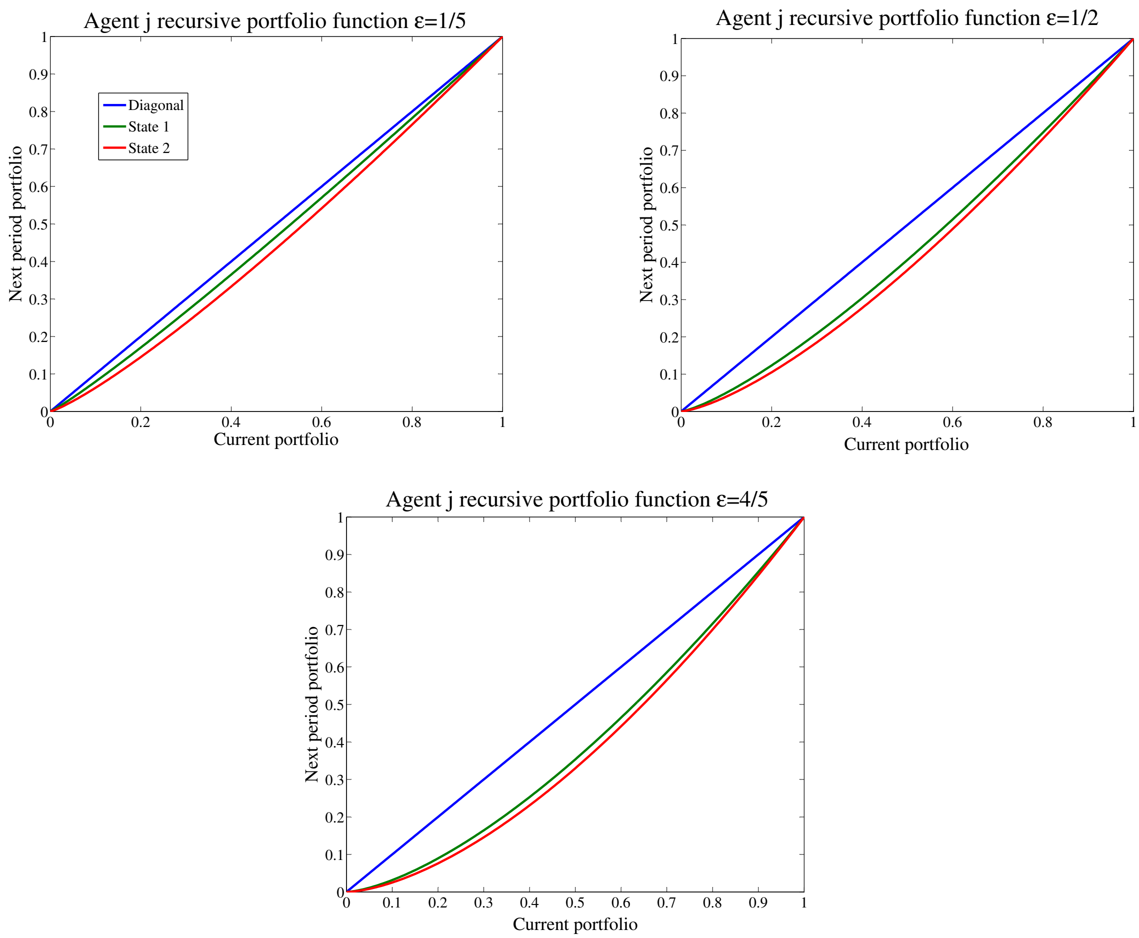

Figure 3: Graph of the policy function $\bar{\theta}^j \mapsto \hat{\theta}^j(\bar{\theta}^j, \bar{\theta}^j, \hat{q}, \hat{\theta})$ for $\epsilon \in \{1/5, 1/2, 4/5\}$.

news tend to become overpriced and have low average returns afterwards. In other words, assets with strings of good performance receive extremely high valuations, and these valuations, on average, return to the mean. Barberis et al. (1998) also relates to another phenomenon documented in psychology, namely conservatism, defined as the slow updating of models in the face of new evidence.[15] The underreaction evidence in particular is consistent with conservatism.

In Barberis et al. (1998), there is one representative investor and one asset with earnings following an i.i.d. process. However, the investor is not aware this law governing exogenous uncertainty. Rather, he believes that dividends moves between two states or regimes. In the first state, earnings are mean-reverting. In the second state, they trend, i.e., are likely to rise further after an increase. The transition probabilities between the two regimes, as well as the statistical properties of the earnings process in each one of them, are fixed in the investor's mind. At each period, the investor observes earnings, and uses this information to update his beliefs about which state he is in. Through his updating, the investor is Bayesian, although his model of the earnings process is inaccurate. Specifically, when a positive earnings surprise is followed by another positive surprise, the investor raises the likelihood that he is in the trending regime. Whereas, when a positive surprise is followed by a negative surprise, the investor raises the likelihood that he is in the mean-reverting regime. Roughly speaking, agents have beliefs under which the dividends are not necessarily perfectly correlated with the states of nature such as in the previous section,[16] and are related to them through a Markov Switching law.

The difference between Barberis et al. (1998) and the example given in this section, is that we consider heterogeneous agents, with one of them having Rational Expectations. Moreover, the exogenous stochastic process is governed by a Markov law. We find numerical results which gives evidence that agents with Markov Switching beliefs may survive in economies with incomplete markets.

The returns on the Lucas Tree Model at each period $t \in \mathbb{N}$ are defined by $r_t = d_t/q_t^a$ where d_t is the dividend on the period t and q_t^a is the asset price at period t.

Writing $\dot{s}_t = (\bar{\theta}_{t-1}, z_t, d_t)$ and $\dot{z}_t = (z_t, d_t) \in \dot{Z} := Z \times D$ for all $t \in \mathbb{N}$, then agents' problem is defined by

$$v^j(\theta^j_-, \dot{z}_1, \boldsymbol{q}) = \sup \left\{ u^j(\boldsymbol{c}^j_1(\dot{z}_1)) + \sum_{t \geq 2} \int_{\dot{Z}^t_2} \beta^{t-1} u^i(\boldsymbol{c}^i_t(\dot{z}_1, \dot{z}^t_2)) \mu^i_t(\dot{z}_1, d\dot{z}^t_2) \right\}$$

over all plans $\{\boldsymbol{c}^j_t, \boldsymbol{\theta}^j_t\}_{t \in \mathbb{N}}$ such that

$$\boldsymbol{q}^c_t(\dot{z}^t)\boldsymbol{c}^j_t(\dot{z}^t) + \boldsymbol{q}^a_t(\dot{z}^t)\boldsymbol{\theta}^j_t(\dot{z}^t) \leq (\boldsymbol{q}^a_t(\dot{z}^t) + \boldsymbol{q}^c_t(\dot{z}^t) d_t)\boldsymbol{\theta}^j_{t-1}(\dot{z}^t)$$

for all $\dot{z}^t \in \dot{Z}^t$ and all $t \in \mathbb{N}$.

In the recursive approach, writing $\dot{s} = (s, d) = (\bar{\theta}, z, d)$ and $\dot{z} = (z, d)$, the Bellman equation is given by

$$v^j(\theta^j_-, \dot{s}, \hat{q}, \hat{\theta}) = \max \left\{ u(c^j) + \beta \int_{\dot{Z}} v^j(\theta^j, (\hat{\theta}(s), \dot{z}'), \hat{q}, \hat{\theta}) \lambda^j(\dot{z}, d\dot{z}') \right\}$$

[15] See Edwards (1968) for more details.

[16] The results of previous sections are applied here. Indeed, it is enough to include the dividends as an additional variable within S.

over all (c^j, θ^j) such that

$$\hat{q}^c(s)c^j + \hat{q}^a(s)\theta^j \leq (\hat{q}^a(s) + \hat{q}^c(s)d)\theta^j_- + \hat{q}^c(s)e^j(z)$$

where λ^j is defined as the Markov switching transition. More formally, consider $Z = \{z_1, z_2\}$ and $D = \{d_l, d_h\}$ as the lower and higher level of dividends. Let $\lambda^j : Z \times D \to \text{Prob}(Z \times D)$ be given by

$$\lambda^j(z,d)(\{z'\},\{d'\}) = \lambda^j_Z(z)(\{z'\})\lambda^j_D(z,d)(\{d'\}) \text{ for all } (z',d') \in Z \times D$$

where $\lambda^j_Z : Z \to \text{Prob}(Z)$ and $\lambda^j_D : Z \times D \to \text{Prob}(D)$. The belief transition λ^j_Z is described by the matrix

$$\lambda^j_Z := \begin{bmatrix} \lambda^j_Z(z_1)(\{z_1\}) & \lambda^j_Z(z_1)(\{z_2\}) \\ \lambda^j_Z(z_2)(\{z_1\}) & \lambda^j_Z(z_2)(\{z_2\}) \end{bmatrix} = \begin{bmatrix} \nu & 1-\nu \\ 1-\nu & \nu \end{bmatrix} \tag{8}$$

forecasting the transition law of the states of nature. The beliefs on the realization of next period dividends are described by the two following law of transitions.

Let $\pi^j_l < 1/2 < \pi^j_h$ be priors about the dividends $\{d_l, d_h\}$ with $d_l < d_h$. On the state 1 the transition $\lambda^j_D(z_1, \cdot)$ is given by the matrix :

$$\lambda^j_D(z_1, \cdot) = \begin{bmatrix} \lambda^j_D(z_1,d_l)(\{d_l\}) & \lambda^j_D(z_1,d_l)(\{d_h\}) \\ \lambda^j_D(z_1,d_h)(\{d_l\}) & \lambda^j_D(z_1,d_h)(\{d_h\}) \end{bmatrix} = \begin{bmatrix} \pi^j_l & 1-\pi^j_l \\ 1-\pi^j_l & \pi^j_l \end{bmatrix}.$$

Notice that on the matrix $\lambda^j_D(z_1, \cdot)$ the level of dividends are likely to be reversed.

On the state 2 the transition $\lambda^j_D(z_2, \cdot)$ is given by the matrix:

$$\lambda^j_D(z_2, \cdot) = \begin{bmatrix} \lambda^j_D(z_2,d_l)(\{d_l\}) & \lambda^j_D(z_2,d_l)(\{d_h\}) \\ \lambda^j_D(z_2,d_h)(\{d_l\}) & \lambda^j_D(z_2,d_h)(\{d_h\}) \end{bmatrix} = \begin{bmatrix} \pi^j_h & 1-\pi^j_h \\ 1-\pi^j_h & \pi^j_h \end{bmatrix}.$$

Notice that on the matrix $\lambda^j_D(z_2)$ the level of dividends are likely to be followed by the same level of previous period dividends. We simulate the model for several true probabilities $p = (p_1, 1-p_1)$ with $p_1 = 0.2, 0.4, 0.41, 0.5$ and 0.8 supposing that $\nu = p_1$, that is, agent j believes that the exogenous uncertainty is a Markov Process with transition λ^j_Z given by (8). We assume that λ^j_Z is the true transition of the economy. Moreover, we fix the parameter $\pi^j_l = 0.3$ and $\pi^j_h = 0.7$. The result is plotted on Figure 4. Notice that the blue line corresponding to the parameter $p_1 = 0.407$ is close to the steady state equilibrium in which there is no trade. In this case all agents survive.

The intuition of this result is that when the probability of state one is high then the asset has lower expected dividends under the true beliefs. Moreover, on state one type j expects that the probabilities are likely to be reversed if $\nu = p_1 = 0.8$. This implies that the asset would have higher next period expected dividends under their beliefs than the true expected dividends anticipated by agents with Rational Expectations. Notice that when the true probability of state one is low then agents with Rational Expectations dominate the market.

Numerical Result 4.4. *If* $p_1 < 0.410$ *then agents with Markov switching beliefs is driven out of the market. If* $p_1 > 0.410$ *agents with Markov switching beliefs dominate the market.*

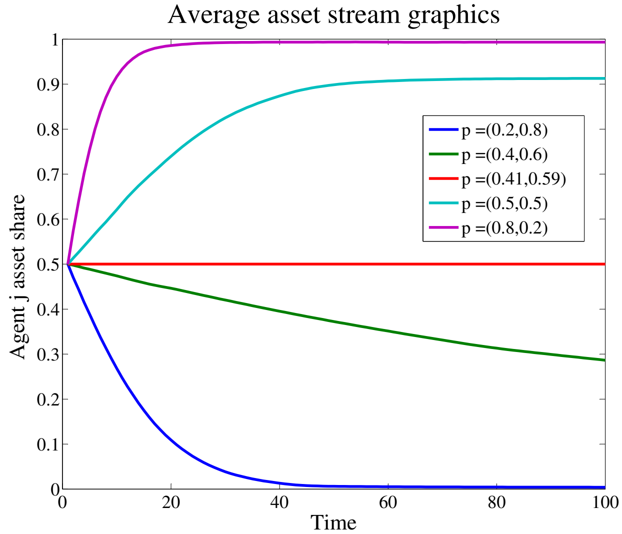

Figure 4: Graph of $\sum_{r\leq 5000}\boldsymbol{\theta}_t^j(z_t)/5000$ for $r = 5000$ trajectories of $(z_1, z_2, ..., z_T)$, $T = 400$ and $\lambda^k(z) = p$ for all $s \in \{z_1, z_2\}$.

4.3 Noise traders

Despite the recognition of the abundance of noise traders in the market, economists feel safe ignoring them in most discussions of asset price formation. The argument against the importance of noise traders for price formation has been forcefully made by Friedman (1953). The author points out that irrational investors are met in the market by rational arbitrageurs who trade against them and, in the process, drive prices close to fundamental values. Consequently, noise traders cannot affect prices too much and, even if they can, will not do so for long.

Shleifer and Summers (1990), De Long et al. (1990) and De Long et al. (1991) elucidate that noise traders can dominate the market for cases where the asset has low returns relative to the risk-free asset. They present a model in which noise traders have incorrect expectations about the variance of the return. Under such expectations, noise traders who by hypothesis do not affect prices, can earn higher expected returns than rational investors with similar risk aversion. Moreover, such noise traders can come to dominate the market, that is, the probability that they eventually have a high share of total wealth is close to one. Preferences are based on the maximization of a constant relative risk aversion approximate utility. The main limitation of De Long et al.'s result is that it does not allow noise traders to affect prices. Furthermore, sophisticated investors do not have preferences embodying consumption strategies. Asset prices are not derived from the market clearing conditions, but from the direct maximization of an unconstrained objective function that directly depends on current and next period asset prices. The example developed here follows the ideas given in De Long et al. (1990). However, we consider the noise traders dealing with traders with rational expectations in an environment of general equilibrium as in Magill and Quinzii (1994) without short sales and, therefore, with equilibrium prices given endogenously.

Let $p = (p_1, p_2, p_3)$ be the true probability. In this case, noise traders have beliefs p^j with correct

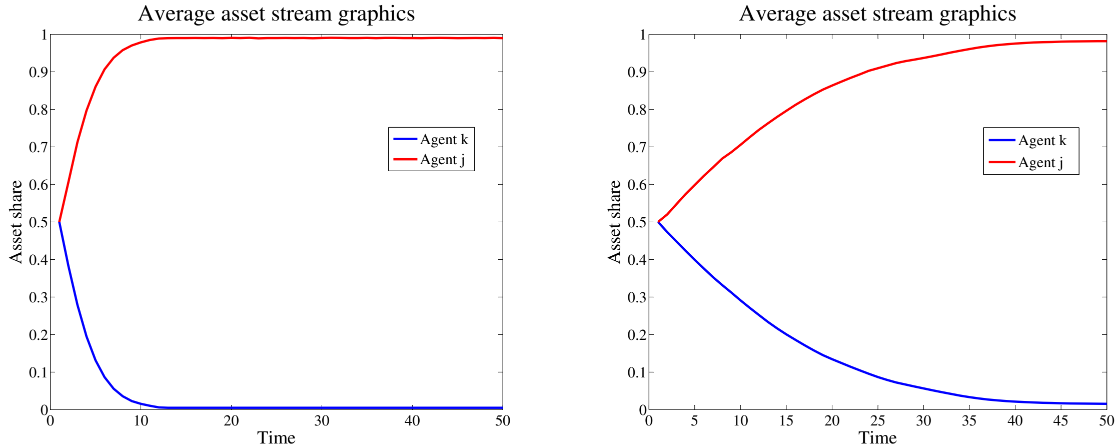

Figure 5: Graph of $\sum_{r\leq 5000}\boldsymbol{\theta}^j_t(\hat{z}_t(\omega))/5000$ for $t = 5000$ trajectories of $(z_1, z_2, ..., z_T)$, $T = 100$, $var^j(d) = 1/5$ on the left $var^j(d) = 2/5$ on the right.

mean $E_p[d]$ and incorrect variance σ^j, that is

$$E^j[d] = \sum_{k\leq 3} p^j_k d(z_k) = E[d] = \sum_{k\leq 3} p_k d(z_k)$$

$$\sigma^j = E_{p_j}[(d - E[d])^2] = \sum_{k\leq 3} p^j_k (d(z_k) - E[d])^2$$

$$\sigma = E_p[(d - E[d])^2] = \sum_{k\leq 3} p_k (d(z_k) - E[d])^2$$

with $\sigma^j \neq \sigma$. Agents with Rational Expectations know the true variance σ and the true mean $E[d]$. The results are illustrated in the Figures 5 and 6.

Numerical Result 4.5. *Noise traders dominate the market if $\sigma^j < \sigma$. If $\sigma^j > \sigma$ they are driven out of the market.*

Remark 4.2. This example suggests that markets select for agents with beliefs under which the induced distribution of dividends is second order stochastic dominant in relation to others. In the case of small σ, the intuition of this statement comes from the fact that agents with rational expectations offer a price at a level above the maximum that noise traders are willing to pay. In other words, agents with rational expectations anticipate that the asset has a low risk-return ratio and therefore, offer a higher price for it.

5 Conclusion

All simulations suggest that incomplete markets favor agents with rational expectations when the expected asset return is higher in relation to the risk-free asset. In this case, agents with rational expectations drive the other agents out of market because they are willing to pay a price at a higher level. In

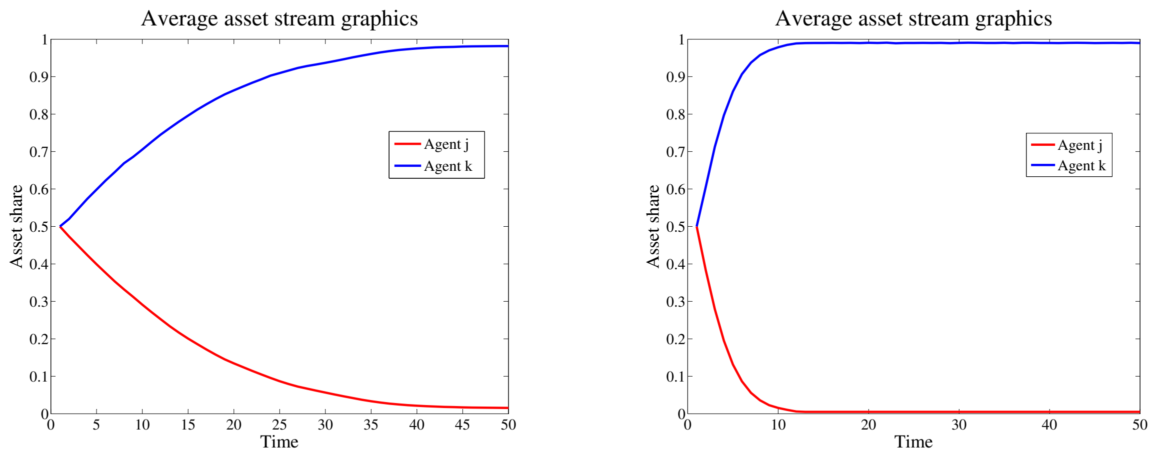

Figure 6: Graph of $\sum_{r\leq 5000}\boldsymbol{\theta}_t^j(\hat{z}_t(\omega))/5000$ for $t = 5000$ trajectories of $(z_1, z_2, ..., z_T)$, $T = 100$, $var^j(d) = 3/5$ on the left $var^j(d) = 4/5$ on the right.

other words, agents with rational expectations anticipate that the asset has a lower risk compared to its return, and therefore, offer a higher price for it. In the examples where agents have ambiguity aversion with ϵ-contamination, all non-rational agents are driven out of the market. In the case of noise traders, markets select for agents with priors in which the induced distribution of the dividends is second order stochastic dominant compared to others. Therefore, markets where assets have lower volatility can favor agents with rational expectations in all simulations.

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6 Appendix

Lemma 6.1. *Suppose that the probability transition rules satisfy for each $A \in \mathscr{Z}$ the relation $\lambda^i(\hat{z}_t(\omega), A) = \mathbb{P}^i(\hat{z}_{t+1} \in A|\mathscr{F}_t)(\omega)$ with $\mathbb{P}^i$ probability one where $\mathscr{F}_t = \sigma(\hat{z}_t)$. That is, $\lambda^i(\hat{z}_t(\cdot), \cdot)$ is a conditional distribution of $\hat{z}_{t+1}$ given $\hat{z}_t$ where $\{\hat{z}_t : \Omega \to S\}_{t\in\mathbb{N}}$ is a stochastic process defined on a probability space $(\Omega, \Sigma, \mathbb{P}^i)$. Then*

$$\mu_t^i(\hat{z}_1(\omega), A_2 \times \cdots \times A_t) = \mathbb{P}^i((\hat{z}_2, ..., \hat{z}_t) \in A_2 \times \cdots \times A_t|\mathscr{F}_1)(\omega).$$

Proof: Theorem 34.5 in Billingsley (1968) states that if $f : Z \to \mathbb{R}$ is a measurable function and $\nu^i : \Omega \times \mathscr{Z} \to [0,1]$ is a conditional distribution of $\hat{z}_2 : \Omega \to Z \subset \mathbb{R}^n$ given $\hat{z}_1 : \Omega \to Z \subset \mathbb{R}^n$ then there exists $\Omega' \in \Sigma$ with $\mathbb{P}^i(\Omega') = 1$ and such that

$$E^i[f(\hat{z}_2(\cdot))|\mathscr{F}_1](\omega) = \int_Z f(z)\nu^i(\omega, dz) \text{ for all } \omega \in \Omega' \in \Sigma. \tag{9}$$

Write $W_r = \{\omega \in \Omega : \hat{z}_r(\omega) \in A_r\}$, $I_{W_r} : \Omega \to \{0, 1\}$ the indicator function for $r = 1, 2, 3$. Therefore, $I_{A_r}(\hat{z}_r(\omega)) = I_{W_r}(\omega)$ for all $\omega \in \Omega$ and $r = 1, 2, 3$. Indeed, $\hat{z}_r(\omega) \in A_r$ if and only if $\omega \in W_r$. Consider the function $f_r(z) = I_{A_r}(z)\lambda^i(z, A_{r+1})$ for all $z \in Z$ and $r = 1, 2, 3$. Pick $W_1 \in \mathscr{F}_1$ arbitrarily. Using f_r in (9) we get

$$\begin{aligned}
\int_{W_1} \mu_3^i(\hat{z}_1(\omega), A_2 \times A_3)\mathbb{P}^i(d\omega) &= \int_{W_1} \left(\int_{A_2} \int_{A_3} \lambda^i(z_2, dz_3)\lambda^i(\hat{z}_1(\omega), dz_2) \right) \mathbb{P}^i(d\omega) \\
&= \int_{W_1} \int_{Z_2} I_{A_2}(z_2)\lambda^i(z_2, A_3)\lambda^i(\hat{z}_1(\omega), dz_2)\mathbb{P}^i(d\omega) \\
&= \int_{W_1} E^i[I_{W_2}(\cdot)\lambda^i(\hat{z}_2(\cdot), A_3)|\mathscr{F}_1](\omega)\mathbb{P}^i(d\omega) \\
&= \int_{W_1} I_{W_2}(\omega)\lambda^i(\hat{z}_2(\omega), A_3)\mathbb{P}^i(d\omega) \\
&= \int_{W_1 \cap W_2} \left(\int_{Z_3} I_{A_3}(z_3)\lambda^i(\hat{z}_2(\omega), dz_3) \right) \mathbb{P}^i(d\omega) \\
&= \int_{W_1 \cap W_2} E^i(I_{W_3}|\mathscr{F}_2)(\omega)\mathbb{P}^i(d\omega) \\
&= \int_{W_1 \cap W_2} I_{W_3}(\omega)\mathbb{P}^i(d\omega) \\
&= \mathbb{P}^i(W_1 \cap W_2 \cap W_3)
\end{aligned}$$

where from first to the second equation we use (9) and that $\nu^i = \lambda^i(\hat{z}_1(\cdot), \cdot) = \lambda^i(\hat{z}_2(\cdot), \cdot)$. From the second to the fifth equation we use the definition of conditional expectation and that $W_\tau \in \sigma(\hat{z}_\tau)$ for all $\tau \in \mathbb{N}$. Therefore,

$$\int_{W_1} \mu_3^i(\hat{z}_1(\omega), A_2 \times A_3)\mathbb{P}^i(d\omega) = \mathbb{P}^i(W_1 \cap (\hat{z}_2 \in A_2) \cap (\hat{z}_3 \in A_3)) \text{ for all } W_1 \in \mathscr{F}_1.$$

By the definition of conditional probability, (Billingsley, 1968) this is the same to state that there exists $\Omega' \in \Sigma$ with $\mathbb{P}^i(\Omega') = 1$ and such that

$$\mu_3^i(\hat{z}_1(\omega), A_2 \times A_3) = \mathbb{P}^i((\hat{z}_2, \hat{z}_3) \in A_2 \times A_3|\mathscr{F}_1)(\omega) = \mathbb{P}^i((\hat{z}_2, \hat{z}_3) \in A_2 \times A_3|\hat{z}_1)(\omega)$$

for all $\omega \in \Omega'$. For $r > 2$ the arguments are analogous.

□

Theorem 6.2. *If $(\hat{c}, \hat{\theta}, \hat{q})$ is a recursive equilibrium then its implemented process $\{\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t\in\mathbb{N}}$ starting from $\bar{\theta} \in \overline{\Theta}$ is a sequential equilibrium of the economy with initial asset holdings $\bar{\theta} \in \overline{\Theta}$.*

Proof: It is sufficient to prove that agent i's choices $\{\hat{\boldsymbol{c}}_t^i, \hat{\boldsymbol{\theta}}_t^i\}_{t\in\mathbb{N}}$ are optimal given the prices $\{\hat{\boldsymbol{q}}_t\}_{t\in\mathbb{N}}$ since, by definition, the recursive equilibrium satisfies all market clearing conditions. Fix $s = (\bar{\theta}, z_1)$ and define $z_2^t = (z_2, z_3, ..., z_t)$. Let $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{F}^i(\bar{\theta}, z_1, \hat{\boldsymbol{q}})$ be an arbitrary feasible plan and write for $n \geq 2$

$$\boldsymbol{U}_n^i(\boldsymbol{c}^i, z_1) = u^i(\boldsymbol{c}_1^i(z_1)) + \sum_{t=2}^{n} \int_{Z_2^t} \beta^{t-1} u^i(\boldsymbol{c}_t^i(z_1, z_2^t))\mu_t^i(z_1, dz_2^t). \tag{10}$$

Therefore, omitting the variables $\hat{q}, \hat{\theta}$ to simplify we get

$$\begin{aligned} v^i(\bar{\theta}^i, s) &\overset{def}{=} \sup\left\{u^i(c^i) + \beta \int_{Z_2} v^i(\theta^i, \hat{\theta}(s_1), z_2)\lambda^i(z_1, dz_2)\right\} \\ &\geq u^i(\boldsymbol{c}_1^i(z_1)) + \beta \int_{Z_2} v^i(\boldsymbol{\theta}_1^i(z_1), \hat{\theta}(s_1), z_2)\lambda^i(z_1, dz_2). \end{aligned} \tag{11}$$

where the sup in the first equation is over all $(c^i, \theta^i) \in B^i(\bar{\theta}^i, z_1, \hat{q}(s_1))$. The inequality above comes from the fact that $(\boldsymbol{c}^i, \boldsymbol{\theta}^i)$ is feasible.[17] Therefore $(\boldsymbol{c}_1^i(z_1), \boldsymbol{\theta}^i(z_1)) \in B^i(\bar{\theta}^i, z_1, \hat{\boldsymbol{q}}_1(z_1)) = B^i(\bar{\theta}^i, z_1, \hat{q}(\bar{\theta}, z_1))$ by the price recursive relation (7) from Definition 3.4 in Section 3. Using the Bellman equation again, we get for each $z_2 \in Z$

$$\begin{aligned} v^i(\boldsymbol{\theta}_1^i(z_1), \hat{\theta}(s_1), z_2) &= \sup\left\{u^i(c^i) + \beta \int_{Z_3} v^i(\theta^i, \hat{\theta}(\hat{\theta}(s_1), z_2), z_3)\lambda^i(z_2, dz_3)\right\} \\ &\geq u^i(\boldsymbol{c}_2^i(z^2)) + \beta \int_{Z_3} v^i(\boldsymbol{\theta}_2^i(z^2), \hat{\theta}(\hat{\theta}(s_1), z_2), z_3)\lambda^i(z_2, dz_3). \end{aligned}$$

where the sup in the first equation is over all $(c^i, \theta^i) \in B^i(\boldsymbol{\theta}_1^i(z_1), z_2, \hat{q}(\hat{\theta}(s_1), z_2))$. The inequality above comes from the fact that $(\boldsymbol{c}^i, \boldsymbol{\theta}^i)$ is feasible and hence $(\boldsymbol{c}_2^i(z^2), \boldsymbol{\theta}_2^i(z^2)) \in B^i(\boldsymbol{\theta}_1^i(z_1), z_2, \boldsymbol{q}_2(z^2)) = B^i(\boldsymbol{\theta}_1^i(z_1), z_2, \hat{q}(\hat{\theta}(s_1), z_2))$ for all $z_2 \in Z$. Indeed, the recursive relation (6) from Definition 3.4 implies that $\hat{\theta}(s_1) = \hat{\boldsymbol{\theta}}_1(z_1)$ and the price recursive relation (7) implies that $\hat{q}(\hat{\theta}(s_1), z_2) = \hat{q}(\hat{\boldsymbol{\theta}}_1(z_1), z_2) = \hat{\boldsymbol{q}}_2(z^2)$. Replacing the previous inequality[18] of $v^i(\boldsymbol{\theta}^i(z_1), \hat{\theta}(s_1), z_2)$ in (11) then

$$\begin{aligned} v^i(\bar{\theta}^i, s_1) &\geq u^i(\boldsymbol{c}_1^i(z_1)) + \beta \int_{Z_2} u^i(\boldsymbol{c}_2^i(z^2))\lambda^i(z_1, dz_2) \\ &+ \beta^2 \int_{Z_2}\int_{Z_3} v^i(\boldsymbol{\theta}_2^i(z^2), \hat{\theta}(\hat{\theta}(s_1), z_2), z_3)\lambda^i(z_2, dz_3)\lambda^i(z_1, dz_2) \\ &= u^i(\boldsymbol{c}_1^i(z_1)) + \beta \int_{Z_2} u^i(\boldsymbol{c}_2^i(z^2))\mu_2^i(z_1, dz_2) \\ &+ \beta^2 \int_{Z_2^3} v^i(\boldsymbol{\theta}_2^i(z^2), \hat{\theta}(\hat{\theta}(s_1), z_2), z_3)\mu_3^i(z_1, dz_2^3) \\ &= \boldsymbol{U}_2^i(\boldsymbol{c}^i, z_1) + \beta^2 \int_{Z_2^3} v^i(\boldsymbol{\theta}_2^i(z^2), \hat{\boldsymbol{\theta}}_2(z^2), z_3)\mu_3^i(z_1, dz_2^3) \end{aligned}$$

[17] That is, $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{F}^i(\bar{\theta}_0, z_1, \boldsymbol{q})$.

[18] See Stokey and Lucas Chapter 9 for more details about the composition of the probability transitions $\{\lambda^i\}_{i\in\mathcal{I}}$.

where in the last equality we use that $\hat{\theta}(\hat{\theta}(s_1), z_2) = \hat{\theta}(\hat{\boldsymbol{\theta}}_1(z_1), z_2) = \hat{\boldsymbol{\theta}}_2(z^2)$ and the equation (10). It follows from induction on n that

$$v^i(\bar{\theta}^i, s_1) \geq \boldsymbol{U}_n^i(\boldsymbol{c}^i, z_1) + \beta^n \int_{Z_2^{n+1}} v^i(\boldsymbol{\theta}_n^i(z^n), \hat{\boldsymbol{\theta}}_n(z^n), z_{n+1}) \mu_{n+1}^i(z_1, dz_2^{n+1}).$$

Taking the limit and using that v^i is bounded we have $v^i(\bar{\theta}^i, s_1) \geq \boldsymbol{U}^i(\boldsymbol{c}^i, z_1)$ for all $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{F}^i(\bar{\theta}^i, z_1, \hat{\boldsymbol{q}})$ since $(\boldsymbol{c}^i, \boldsymbol{\theta}^i)$ was chosen arbitrarily.

Let $\tilde{\theta}^i : \Theta^i \times S \times \widehat{Q}^\circ \times \widehat{\Theta} \to \Theta^i$ and $\tilde{c}^i : \Theta^i \times S \times \widehat{Q}^\circ \times \widehat{\Theta} \to C^i$ be the policy functions according to definition 3.2. Consider the plan[19] $\{\tilde{\boldsymbol{c}}_t^i, \tilde{\boldsymbol{\theta}}_t^i\}_{t\in\mathbb{N}}$ defined recursively for each $i \in \mathcal{I}$ as

$$\tilde{\boldsymbol{c}}_t^i(z^t) \in \tilde{c}^i(\tilde{\boldsymbol{\theta}}_{t-1}^i(z^{t-1}), (\tilde{\boldsymbol{\theta}}_{t-1}(z^{t-1}), z_t), \hat{q}, \hat{\theta}) = \hat{c}^i(\tilde{\boldsymbol{\theta}}_{t-1}(z^{t-1}), z_t) \tag{12}$$

$$\tilde{\boldsymbol{\theta}}_t^i(z^t) \in \tilde{\theta}^i(\tilde{\boldsymbol{\theta}}_{t-1}^i(z^{t-1}), (\tilde{\boldsymbol{\theta}}_{t-1}(z^{t-1}), z_t), \hat{q}, \hat{\theta}) = \hat{\theta}^i(\tilde{\boldsymbol{\theta}}_{t-1}(z^{t-1}), z_t) \tag{13}$$

where $\tilde{\boldsymbol{\theta}}_0 = \bar{\theta}$. Since by hypothesis, $\{\hat{\boldsymbol{c}}_t, \hat{\boldsymbol{\theta}}_t, \hat{\boldsymbol{q}}_t\}_{t\in\mathbb{N}}$ is a stream implemented by $(\hat{c}, \hat{\theta}, \hat{q})$, then (13) implies that $\tilde{\boldsymbol{\theta}}_1^i(z_1) = \hat{\theta}^i(\bar{\theta}, z_1) = \hat{\boldsymbol{\theta}}_1^i(z_1)$ for all $i \in \mathcal{I}$ and thus $\tilde{\boldsymbol{\theta}}_1(z_1) = \hat{\boldsymbol{\theta}}_1(z_1)$. Using relation (6) recursively, we get $\tilde{\boldsymbol{\theta}}_t(z^t) = \hat{\boldsymbol{\theta}}_t(z^t)$ and hence $\tilde{\boldsymbol{c}}_t(z^t) = \hat{\boldsymbol{c}}_t(z^t)$ for all $t \in \mathbb{N}$. Furthermore, Equation (7) implies that $\hat{q}(\tilde{\boldsymbol{\theta}}_{t-1}(z^{t-1}), z_t) = \hat{q}(\hat{\boldsymbol{\theta}}_{t-1}(z^{t-1}), z_t) = \hat{\boldsymbol{q}}_t(z^t)$. Since the policy correspondences $\tilde{c}^i$ and $\tilde{\theta}^i$ satisfy (5) then

$$(\tilde{\boldsymbol{c}}_t^i(z^t), \tilde{\boldsymbol{\theta}}_t^i(z^t)) \in B^i(\tilde{\boldsymbol{\theta}}_{t-1}^i(z^{t-1}), z_t, \hat{q}(\tilde{\boldsymbol{\theta}}_{t-1}(z^{t-1}), z_t))$$

and hence $(\tilde{\boldsymbol{c}}_t^i(z^t), \tilde{\boldsymbol{\theta}}_t^i(z^t)) \in B^i(\tilde{\boldsymbol{\theta}}_{t-1}^i(z^{t-1}), z_1, \hat{\boldsymbol{q}}_t(z^t))$ for all $z^t \in Z^t$ and all $t \in \mathbb{N}$, that is, $(\hat{\boldsymbol{c}}^i, \hat{\boldsymbol{\theta}}^i) = (\tilde{\boldsymbol{c}}^i, \tilde{\boldsymbol{\theta}}^i) \in \boldsymbol{F}^i(\bar{\theta}^i, z_1, \hat{\boldsymbol{q}})$. The construction of the plan $(\tilde{\boldsymbol{c}}^i, \tilde{\boldsymbol{\theta}}^i)$ implies that all inequalities of the above arguments must bind. Thus $v^i(\bar{\theta}^i, s_1) = \boldsymbol{U}^i(\tilde{\boldsymbol{c}}^i, z_1) = \boldsymbol{U}^i(\hat{\boldsymbol{c}}^i, z_1)$ and hence $(\hat{\boldsymbol{c}}^i, \hat{\boldsymbol{\theta}}^i) \in \delta^i(\bar{\theta}^i, z_1, \hat{\boldsymbol{q}})$. Therefore, $\{\hat{\boldsymbol{c}}_t, \hat{\boldsymbol{\theta}}_t, \hat{\boldsymbol{q}}_t\}_{t\in\mathbb{N}}$ is a sequential equilibrium for the economy $\mathcal{E}$.

□

[19] When the argmax of the Bellman equation (3) is a correspondence, then the proof is analogous using the Measurable Maximum Theorem and choosing a suitable selector. See Aliprantis and Border (1999) for more detail.

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